Fourier Transform: $f(x)$.

$$\hat{f}(w) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)\cdot e^{-iwx}\, dw.$$

$$f(x) = \int_{-\infty}^{+\infty} \hat{f}(w) e^{iwx} dw.$$

$f(x)$ $\qquad\qquad$ $\hat{f}(w)$

$f'(x)$ $\qquad\qquad$ $iw\hat{f}(w)$

$f * g$ $\qquad\qquad$ $\hat{f}\cdot\hat{g}$

$$f*g(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\bar{x})\cdot g(x-\bar{x})\, d\bar{x}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x-\bar{x})\, g(\bar{x})\, d\bar{x}$$

$f(x) = e^{-\alpha x^2}$ $\qquad\qquad$ $\hat{f}(w) = \frac{1}{\sqrt{4\pi\alpha}} e^{-w^2/4\alpha}$

$\sqrt{\frac{\pi}{\beta}}\, e^{-x^2/4\beta}$ $\qquad\qquad$ $e^{-\beta w^2}$

$u_t = k u_{xx}$ $\quad$ $u(x,0) = f(x)$.

$\hat{u}_t = k(iw)^2 \hat{u}$.

$\hat{u} = e^{-kw^2 t} C(w)$.

$t=0$, $\hat{u} = \hat{f}(w) \Rightarrow C(w) = \hat{f}(w)$

$\hat{u} = e^{-kw^2 t} \cdot \hat{f}(w)$.

$u(x,t) = f(x) * \text{IFT of } e^{-ktw^2}$.

$= f(x) * \sqrt{\frac{\pi}{kt}} e^{-\frac{x^2}{4kt}}$.

$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sqrt{\frac{\pi}{kt}} f(\bar{x}) \cdot e^{-\frac{(x-\bar{x})^2}{4kt}} d\bar{x}$.

wave equation:

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(x,0) = f(x) \\ u_t(x,0) = 0 \end{cases}$$

$$\hat{u}_{tt} = c^2(-iw)^2 \hat{u} = -c^2 w^2 \hat{u}.$$

$$\hat{u} = \cos(cwt)\, A(w) + \sin(cwt)\, B(w).$$

$$\hat{u}(w,0) = \hat{f}(w) \Rightarrow A(w) = f(w)$$

$$\hat{u}_t(w,0) = 0 \Rightarrow B(w) = 0$$

$$u(x,t) = f(x) * \text{IFT of } \underbrace{\cos(cwt)}_{\text{something like Dirac Delta function.}}$$

Skip this part if not comfortable with Dirac Delta function.

$$= f(x) * \underbrace{\frac{e^{icwt} + e^{-icwt}}{2}}_{\text{FFT of}}$$

$$= \frac{1}{2}\left(f(x+ct) + f(x-ct)\right)$$

$$u(x,t) = \int_{-\infty}^{+\infty} \hat{f}(w) \cdot \cos(cwt) \cdot e^{-iwx} \, dw$$

$$= \int_{-\infty}^{+\infty} \hat{f}(w) \cdot \frac{e^{icwt} + e^{-icwt}}{2} \cdot e^{-iwx} \, dw$$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} \hat{f}(w) e^{-iw(x+ct)} \, dw + \frac{1}{2}\int_{-\infty}^{+\infty} \hat{f}(w) e^{-iw(x-ct)} \, dw.$$

$$= \frac{1}{2}\left(f(x+ct) + f(x-ct)\right)$$

when there this IFT of cos, sin.

Try to apply direct formula.

Ex:

$$\begin{cases} u_t = c u_x, \\ u(x,0) = f(x) \end{cases}$$

$$\hat{u}_t = (-ic\omega)\,\hat{u}$$

$$u = e^{-ic\omega t}\,\hat{f}(\omega)$$

$$u(x,t) = \int_{-\infty}^{+\infty} \hat{f}(\omega) \cdot e^{-ic\omega t} \cdot e^{-i\omega x}\, d\omega$$

$$= \int_{-\infty}^{+\infty} \hat{f}(\omega) \cdot e^{-i\omega(x+ct)}\, d\omega$$

$$= f(x+ct)$$

Laplace equation

$$\Delta u = 0$$

$$u(x,0) = f(x)$$

$$\lim_{y \to \infty} u(x,y) = 0.$$

$$\hat{u}(w,y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} u(x,y) e^{iwx} \, dx.$$

$$(iw)^2 \hat{u} + \hat{u}_{yy} = 0. \qquad \hat{u}_{yy} = w^2 \hat{u}$$

$$\hat{u} = A(w) \cdot e^{wy} + B(w) e^{-wy}.$$

$$\Rightarrow \quad A(w) = 0 \quad \text{if } w > 0$$

$$B(w) = 0 \quad \text{if } w < 0.$$

$$\hat{u} = A(w)\cdot e^{-|w|y}.$$

$$\hat{u} = \hat{f}(w)\cdot \underline{e^{-|w|y}}$$

$$u(x,y) = f(x) * \text{IFT of } e^{-|w|y}$$

$$\int_{-\infty}^{+\infty} e^{-|w|y}\cdot e^{-iwx}\,dw.$$

$$= \int_{0}^{+\infty} e^{-wy} e^{-iwx}\,dw$$

$$+ \int_{-\infty}^{0} e^{wy} e^{-iwx}\,dw$$

$$= \frac{1}{y+ix} + \frac{1}{y-ix} = \frac{2y}{y^2+x^2}$$

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\bar{x})\cdot \frac{2y}{y^2+(x-\bar{x})^2}\,d\bar{x}$$

Convolution Theorem:

$$\hat{h}(w) = \hat{f}(w) \cdot \hat{g}(w)$$

Then $h(x) =$

$$\int_{-\infty}^{+\infty} \hat{f}(w) \cdot \hat{g}(w) e^{-iwx} \, dw$$

$$= \int_{-\infty}^{+\infty} \frac{1}{2\pi} f(\bar{x}) e^{iw\bar{x}} \, d\bar{x} \; \hat{g}(w) e^{-iwx} \, dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\bar{x}) \hat{g}(w) e^{-iw(x-\bar{x})} \, dw \, d\bar{x}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\bar{x}) \cdot g(x-\bar{x}) \, d\bar{x}$$

$$= f * g$$

Fourier transform of Gaussian.

$$\hat{g}(w) = e^{-\beta w^2}$$

$$g(x) = \int_{-\infty}^{+\infty} e^{-\beta w^2} \cdot e^{-iwx} \, dw$$

$$\frac{dg}{dx} = \int_{-\infty}^{+\infty} (-iw) e^{-\beta w^2} e^{-iwx} \, dw$$

$$\frac{d e^{-\beta w^2}}{dw} = -2\beta w e^{-\beta w^2}$$

$$\frac{dg}{dx} = \int_{-\infty}^{+\infty} \frac{-i}{-2\beta} \frac{d e^{-\beta w^2}}{dw} e^{-iwx} \, dw$$

$$= \frac{i}{2\beta} \cdot - \int_{-\infty}^{+\infty} e^{-\beta w^2} (-ix) e^{-iwx} \, dw.$$

$$= -\frac{x}{2\beta} \int_{-\infty}^{+\infty} e^{-\beta w^2} \cdot e^{-iwx} \, dw$$

$$= -\frac{x}{2\beta} g(x)$$

so $g(x) = e^{-\frac{x^2}{4\beta}} \cdot C$

$$g(0) = C = \int_{-\infty}^{+\infty} e^{-\beta w^2} \, dw$$

$s = \sqrt{\beta} w$

$$= \frac{1}{\sqrt{\beta}} \int_{-\infty}^{+\infty} e^{-s^2} \, ds$$

$$= \frac{\sqrt{\pi}}{\sqrt{\beta}}.$$

so $g(x) = \sqrt{\frac{\pi}{\beta}} \cdot e^{-\frac{x^2}{4\beta}}$