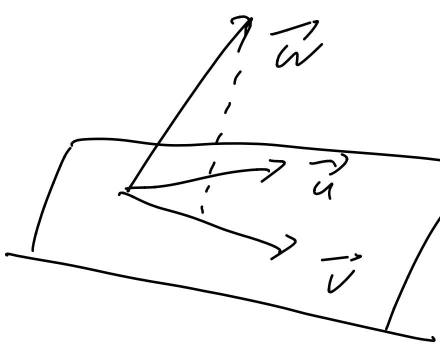
Linear algebra and its applications.

— Lay, Lay, McDonald.

Chapter 6 Othogonality and least squares.

How to find $\mathrm{Proj}_{\text{plane generated by } \vec{u}, \vec{v}} \vec{w}$ projections.?



Goal: · Equilibrium sol'n for heat equations

· higher dimensional heat equations

1D: $\rho \cdot C\, u_t = (K_0\, u_x)_x + Q$.

$$u_t = k\, u_{xx} + Q(x)$$

If ρ, C, K_0 constant, Q does not depend on time.

Intuition: · approach "equilibrium" solutions. as $t \to \infty$.

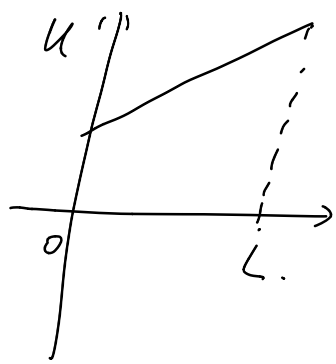
· If $Q(x) = 0$, and insulated boundaries. evens out the initial temp

$$\boxed{\text{Equilibrium solutions. means } u_t = 0.}$$

Ex: $u_t = k\, u_{xx}$.

$u_t = 0 \Rightarrow u_{xx} = 0$.

so $u = ax + b$



Dirichlet:

(BC) $u(0,t) = T_0$, $u(L,t) = T_1$ $\Rightarrow$ $b = T_0$. $a = \dfrac{T_1 - T_0}{L}$

Insulated boundary: $u_x(0,t)=0.$ $u_x(L,t)=0$

$\Rightarrow$ $a=0.$ Still need initial condition $u(x,0)$ to find b.

$H(t) = \int_0^L u\,dx.$ then $\frac{dE}{dt} = 0$

for insulated boundaries.

Why? $$\frac{d}{dt} H(t) = \int_0^L u_t \, dx$$

$$= \int_0^L k\, u_{xx}\, dx$$

$$= k \cdot u_x \Big|_{x=0}^{x=L} = 0$$

Initial condition: $u(x,0) = f(x)$

as $t \to +\infty$, $u(x,t) \to \underline{u(x)} = b.$

equilibrium solution.

$$H(0) = \int_0^L f(x)\,dx$$

$$H(\infty) = \int_0^L b\,dx = b \cdot L$$

$$\Rightarrow b = \frac{1}{L} \cdot \int_0^L f(x)\,dx.$$

Equilibrium solutions with heat source.

$$u_t = u_{xx} + 2$$ heat source Q.

$$u_x(0,t) = 3$$ heat gain/loss at two ends.

$$u_x(L,t) = \alpha$$ (BC)

$$u(x,0) = f(x)$$ (IC)

Equilibrium solution $u(x) = \lim_{t\to+\infty} u(x,t)$

$$u_{xx} = -2 \Rightarrow u_x = -2x + C_1$$

$$\Rightarrow u = -x^2 + C_1 x + C_2.$$

then $u_x(0) = C_1 = 3$

$$u_x(L) = -2L + C_1 = \alpha \Rightarrow \boxed{\alpha = 3-2L}.$$

$$H(t) = \int_0^L u(x,t)\,dt$$

$$H'(t) = \frac{d}{dt}\int_0^L u(x,t)\,dt = \int_0^L u_t\,dt = \int_0^L (u_{xx}+2)\,dt$$

$$= u_x(x,t)\Big|_{x=0}^{x=L} - 2L = 0$$

$$H(0) = \int_0^L f(x)\,dt = H(\infty) = \int_0^L -x^2+3x+C_2\,dx.$$

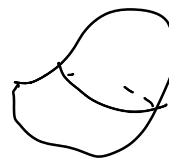
$$\Rightarrow C_2 = \frac{1}{L}\int_0^L f(x)\,dx + \frac{L^2}{3} - \frac{3}{2}L,$$

so equilibrium solution $u(x) = -x^2 + 3x + \frac{1}{L}\int_0^L f(x)\,dx + \frac{L^2}{3} - \frac{3}{2}L.$

Higher dimensional heat equation.

$$E(t) = \int_\Omega c\rho\, u(x,y,t).$$

$$E'(t) = \int_\Omega c\rho\, u_t.$$



Ω.

Heat gain/loss through $\partial\Omega$.

(Notation: $D_{\vec{n}} u = \frac{\partial u}{\partial \vec{n}}$.)

Fourier's law. $\phi = -k_0 \cdot (D_{\vec{n}} u) = -k_0 \cdot \langle \nabla u, \vec{n} \rangle$

$$= -\int_{\partial\Omega} \phi + \int_\Omega Q.$$

$$E'(t) = \int_{\partial\Omega} k_0 \langle \nabla u, \vec{n} \rangle + \int_\Omega Q$$

$$= \int_\Omega \mathrm{div}(k_0 \nabla u) + \int_\Omega Q$$

$$\Rightarrow \rho c\, u_t = \mathrm{div}(k_0 \nabla u) + Q.$$

If (thermal properties) ρ, c, k_0 constants, $Q = 0$

$$u_t = k\,\Delta u.$$