There are Mathematica codes availablable on course website.

Try to observe in which cases the solutions to heat equations converge to equilibrium solutions.

Explain those cases both mathematically and physically.

(Mathematica is avaible for Penn students. search Mathematica Penn)

Equilibrium solutions satisfy Laplace equation.

$$\Delta u = 0. \quad \text{(very important PDE)}$$

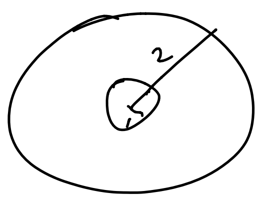
or $\Delta u = f(x, y, t)$ ← (Possion equation)

2D Laplacian in polar coordinates.

$$\boxed{\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.}$$

need to remember this formula.

Ex.



$A = \{(x,y) \in \mathbb{R}^2 \mid 1 < x^2+y^2 < 4\}$

$u(x,y)$.

$$\Delta u = 4, \quad \text{(BC)} \begin{cases} u = 2 & \text{when } r = 1 \\ u = 5 - \log 2 & \text{when } r = 2. \end{cases}$$

(Dirichlet problem for poisson equation).

Guess solution $u(r,\theta) = u(r)$

not depending on θ.

then $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 4$$

$$\Rightarrow \quad (r u')' = 4r$$

$$\Rightarrow \quad r u' = 2r^2 + C_1$$

$$u' = 2r + \frac{C_1}{r}$$

$$u = r^2 + C_1 \log r + C_2.$$

$u(1) = 2$

$u(2) = 5 - \log 2$

$\Rightarrow \quad C_1 = -1, \quad C_2 = 1.$

$u(r,\theta) = r^2 - \log r + 1$

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why no other solutions.

Try to prove uniqueness for Dirichlet problem.

① linearity method.

② Energy method.

① If u_1, u_2 are both solutions to $\Delta u = f$.

D some region. $u|_{\partial D} = g$ (BC)

Consider $v = u_1 - u_2$

then $\begin{cases} \Delta v = 0 . \\ v|_{\partial D} = 0 \end{cases}$

② $E = \int_D \langle \nabla v , \nabla v \rangle$

$= \int_{\partial D} v \cdot \langle \nabla v, \vec{n} \rangle - \int_D v \Delta v$

$= 0$

$\left(\int_D f \Delta g = \int_{\partial D} f \langle \nabla g , \vec{n} \rangle - \int_D \langle \nabla f , \nabla g \rangle \right)$

So $E = \int |\nabla v|^2 = 0$

$\nabla v = \vec{0} \Rightarrow v =$ constant.

Uniqueness for heat equation:

$$u_t = u_{xx} + Q(x)$$

$$\left.\begin{aligned} u(0,t) &= f(t) \\ u(L,t) &= g(t). \end{aligned}\right\} (BC)$$

$$u(x,0) = h(x) \quad (IC).$$

If u_1. u_2 are two solutions.

thus $v = u_1 - u_2$ is a solution

to $\quad v_t = v_{xx}$

$$v(0,t) = v(L,t) = 0$$

$$v(x,0)=0.$$

$$E(t)=\int_0^L v^2(x)\,dx.$$

$$E'(t)=\int_0^L v\,v_t\,dx$$

$$=\int_0^L v\,v_{xx}\,dx$$

$$=-\int_0^L (v_x)^2\,dx$$

$$+\;v\cdot v_x\Big|_{x=0}^{x=L}$$

$$\le 0$$

From IC, $E(0) = 0$.

$$E(t) = \int_0^l v^2 \, dx \geq 0.$$

because $v^2 \geq 0$

$E(t)$ is decreasing, so $E(t) = 0$ for $t \geq 0$

and $\int_0^l v^2 \, dx = 0 \Rightarrow v(x,t) = 0$ for $t \geq 0$.