Linear Operators. (Use linear algebra to solve PDEs)

Ex: $L(u) = u_x$.
$L(u) = \delta u$.
linear operators.

$$L(u+v) = L(u) + L(v)$$

$$L(c \cdot v) = c \cdot L(v)$$

c is a constant

Non-Ex: $L(u) = u \cdot u_x$.

Heat equation is linear.

"$u_t = k u_{xx}$." is linear

means $Lu = u_t - k u_{xx}$ is

linear operator.

Set of solutions "$Lu = 0$"

is a vector space. (Addition of two solutions $u+v$, scalar product $\lambda \cdot u$ are solutions)

General notation:

$Lu = 0$ homogeneous linear equation

$Lu = f$ (f not zero function)

inhomogeneous linear equation.

"Principle of superposition"/linearity:

If u_1, u_2 are solutions to a linear homog. eqn. then so is

$$c_1 u_1 + c_2 u_2 \quad \text{for any } c_1, c_2 \in \mathbb{R}.$$

Seperation of variables ← Most important topic of the class.

1D heat equation

$$u_t = k u_{xx} \qquad (PDE)$$

$$u(0,t) = u(L,t) = 0 \qquad (BC)$$

$$u(x,0) = f(x) \qquad (IC)$$

Idea: Ignore (IC) at first.

Look for soln's

$u(x,t) = \phi(x)\, G(t)$.

$u_t = \phi(x) \cdot G'(t)$

$u_{xx} = \phi''(x)\, G(t)$.

$u_t = k u_{xx}$.

$\Leftrightarrow \quad \phi(x) \cdot G'(t) = \phi''(x)\, G(t)$

$\Leftrightarrow \quad \frac{1}{k} \frac{G'(t)}{G(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda$.

fcn of t.

fcn of x

so λ is a constant.

We get a system of ODEs

$$G'(t) = \lambda \cdot K\, G(t)$$

$$\phi''(x) = \lambda \phi(x).$$

(BC): $u(0,t) = 0 \Rightarrow \phi(0) \cdot G(t) = 0$

$u(L,t) = 0 \Rightarrow \phi(L)\, G(t) = 0.$

$\Rightarrow \phi(0) = \phi(L) = 0.$

(Otherwise $G(t) = 0$ for all t, which gives 0 solution)

$\phi''(x) = -\lambda \phi(x)$ and $\phi(L) = \phi(0) = 0.$

("Eigenvalue problem".
Like matrix equation $Av = \lambda v$)

- What are values λ for which there're no zero soln's
- What are the soln's.

Case 1. $\lambda < 0$.

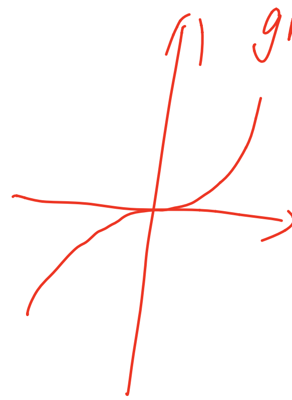
$$\phi''(x) = C_1 \cosh\sqrt{-\lambda}\, x + C_2 \sinh\sqrt{-\lambda}\, x.$$

$$\phi(0) = 0 \Rightarrow C_1 = 0$$

$$\phi(L) = 0 \Rightarrow C_2 \sinh\sqrt{-\lambda}\, L = 0$$

$$\Rightarrow C_2 = 0$$

$$\Rightarrow \phi(x) = 0$$



graph of $\sinh x$

Case 2. $\lambda = 0$, $\phi(x) = ax + b$.

$$\phi(0) = \phi(L) = 0 \Rightarrow \phi(x) = 0$$

Case 3. $\lambda > 0$, $\phi(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$.

$\phi(0) = 0 \Rightarrow C_1 = 0$

$\phi(L) = 0 \Rightarrow \sin\sqrt{\lambda}L = 0$

$\Rightarrow \sqrt{\lambda}L = n\pi$, $\lambda = \dfrac{n^2\pi^2}{L^2}$.

$\phi(x) = \sin\dfrac{n\pi}{L}x$.

Another method of proving $\lambda > 0$ by integration by parts.

$$\begin{cases} \phi'' = -\lambda\phi \\ \phi(0) = \phi(L) = 0 \end{cases}$$

$$\int_0^L \phi \cdot \phi'' dx = -\lambda \int_0^L \phi^2 dx$$

$$\int_0^L \phi\, \phi'' x = \phi \cdot \phi' \Big|_0^L - \int_0^L (\phi')^2 dx$$

So $\lambda = \dfrac{\int_0^L (\phi')^2\,dx}{\int_0^L (\phi)^2\,dx}$

If $\phi \not\equiv 0$, then $\lambda > 0$

On the other hand:

$G'(t) = -k\lambda G(t)$

$\Rightarrow G = e^{-k\lambda t} = e^{-k\frac{n^2\pi^2}{L^2}t}$.

So we find solns

$u(x,t) = \sin\frac{n\pi}{L}x \cdot e^{-k\frac{n^2\pi^2}{L^2}t}$

$n = 1, 2, 3 \ldots$

with IC

$u(x,0) = \sin\frac{n\pi}{L}x$

If $f(x) = \sum_{n=1}^{N} C_n \sin \frac{n\pi}{L} x$.

then we find unique soln

$$u(x,t) = \sum_{n=1}^{N} C_n \sin \frac{n\pi}{L} x \cdot e^{-k\frac{n^2\pi^2}{L^2} t}$$

From Fourier expansion:

any continuous function f such that $f(0) = f(L) = 0$. can be written as.

$$f(x) = \sum_{n=1}^{+\infty} C_n \sin \frac{n\pi}{L} x$$

So general soln:

$$u(x,t) = \sum_{n=1}^{+\infty} C_n \sin \frac{n\pi}{L} x \, e^{-\frac{kn^2\pi^2}{L^2} t}.$$

Question: How to find C_n?

In HW1, If $v_1 \cdots v_n$ are orthogonal basis, i.e. $\langle v_i, v_j \rangle = 0$ when $i \neq j$

$$W = \sum_{i=1}^{n} C_i v_i,$$ then

$$C_m = \frac{\langle W, v_m \rangle}{\langle v_m, v_m \rangle}$$

Similar formula holds for

$$f(x) = \sum_{n=1}^{+\infty} C_n \sin \frac{n\pi}{L} x$$

In HW2:
$$\int_0^L \left(\sin \frac{n\pi}{L} x\right)\left(\sin \frac{m\pi}{L} x\right) dx$$
$$= \begin{cases} 0 & m \neq n \\ \frac{L}{2} & m = n. \end{cases}$$

So $\int_0^L f(x)\cdot(\sin \frac{m\pi}{L}x)\,dx$

$$= \sum_{n=1}^{\infty} \int_0^L C_n \cdot (\sin \frac{n\pi}{L}x \sin \frac{m\pi}{L}x)\,dx$$

$$= C_m \cdot \frac{L}{2}$$

So $C_m = \dfrac{2\int_0^L f(x)\cdot(\sin \frac{m\pi}{L}x)\,dx}{L}$

Define the dot product (inner product) of two functions $f(x), g(x)$ by

$$\langle f, g\rangle = \int_0^L f(x)\cdot g(x)\,dx$$

So $\quad \left\langle \sin\frac{n\pi}{L}x, \sin\frac{m\pi}{L}x \right\rangle = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n \end{cases}$

$$C_m = \frac{\left\langle f, \sin\frac{m\pi}{L}x \right\rangle}{\left\langle \sin\frac{m\pi}{L}x, \sin\frac{m\pi}{L}x \right\rangle}$$