Math 241 Sep 12 Lec 6

Recall
$$u_t = k u_{xx}$$
$$u(0,t) = u(L,t) = 0 \quad (BC)$$
$$u(x,t) = f(x) \quad (IC)$$

$$u = \phi(x)\, G(t).$$

Solve
$$\begin{cases} \phi'' = -\lambda\phi \\ \phi(0) = \phi(L) = 0 \quad (BC). \end{cases} \quad (*)$$

Eigenvalues: $\lambda_n = \left(\frac{n\pi}{L}\right)^2$

Eigenvectors: $\phi_n = \sin\frac{n\pi}{L}x.$

Write $f(x) = \sum_{n=1}^{+\infty} B_n \sin\frac{n\pi}{L}x$

$$B_n = \frac{2}{L}\int_0^L f(x)\cdot\sin\frac{n\pi}{L}x\cdot dx.$$

$$u(x,t) = \sum_{n=1}^{+\infty} B_n \sin\frac{n\pi x}{L}\, e^{-k\frac{n^2\pi^2}{L^2}x}$$

Key: $\int_0^L \phi_n\cdot\phi_m\, dx = 0$ if $m \neq n$.

Proof by (*)

$$\int_0^L \phi_n \, \phi_m'' \, dx = -\phi_n \cdot \phi_m' \Big|_0^L - \int_0^L \phi_n' \phi_m' \, dx$$

$$= \overset{(BC)}{0} - \int_0^L \phi_n' \phi_m' \, dx$$

$$= -\left(\phi_n' \phi_m \Big|_0^L - \int_0^L \phi_n'' \phi_m \, dx \right)$$

$$= \overset{(BC)}{0} + \int_0^L \phi_n'' \phi_m \, dx.$$

So $\displaystyle\int_0^L \phi_n \phi_m'' \, dx = \int_0^L \phi_n'' \phi_m \, dx.$

$\phi_m'' = -\lambda_m \phi_m, \quad \phi_n'' = -\lambda_n \phi_n.$

So $\displaystyle -\lambda_m \int_0^L \phi_n \phi_m \, dx = -\lambda_n \int_0^L \phi_n \cdot \phi_m \, dx.$

$\displaystyle \Rightarrow \int_0^L \phi_n \cdot \phi_m \, dx = 0.$

Non-homogeneous equation with constant Dirichlet condition:

$$u_t = \Delta u + Q(x).$$

$$u(0,t) = T_1, \quad u(L,t) = T_2 \quad (BC)$$

$$u(x,t) = f(x). \quad (IC)$$

Recall in ODE, $u'' + u = f(x)$.

Find one soln, u_0,

then any soln. u_1, $v = u_1 - u_0$,

v satisfies the homogeneous equation

$$v'' + v = 0.$$

In PDE, same method works.

Special solution: equilibrium solution.

$u_0(x,t) = u_0(x)$ (HW 2) (Not satisfying the IC)

Any solution $v(x)$ satisfies.

$$v_t = k v_{xx}.$$

$$v(0,t) = v(L,t) = 0$$

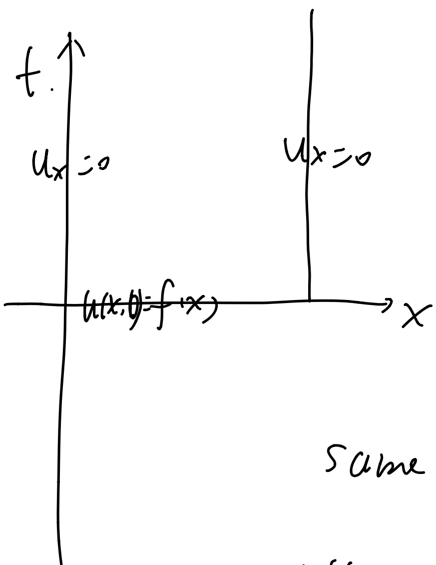
$$v(x,t) = f(x) - u_0(x)$$

Neumann condition.

$$u_t = k u_{xx}$$

$$u_x(0,t) = u_x(L,t) = 0$$

$$u(x,t) = f(x)$$



Ignore $u(x,0) = f(x)$ (IC)

Write $u(x,t) = \phi(x) \cdot G(t)$.

same logic $\Rightarrow$ Eigenvalue problem:

$$\begin{cases} \phi'' = -\lambda \phi \\ \phi'(0) = \phi'(L) = 0. \end{cases}$$

Solns are $\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 0, 1, \ldots$

$$\phi_n = \cos \frac{n\pi}{L} x$$

Othogonality: $$\int_0^L \cos\frac{n\pi}{L}x\cos\frac{m\pi}{L}x = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n \neq 0 \\ L, & m = n = 0. \end{cases}$$

$$f(x) = \sum_{n=0}^{+\infty} A_n \cdot \cos\frac{n\pi x}{L}$$ (Fourier expansion)

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx.$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cdot\cos\frac{n\pi x}{L}\,dx$$

soln: $$u(x) = \sum_{n=0}^{+\infty} A_n \cdot \cos\frac{n\pi x}{L}\cdot e^{-\frac{kn^2\pi^2}{L^2}t}$$

$$= A_0 + \sum_{n=1}^{+\infty} A_n \cos\frac{n\pi x}{L} e^{-\frac{kn^2\pi^2}{L^2}t}.$$