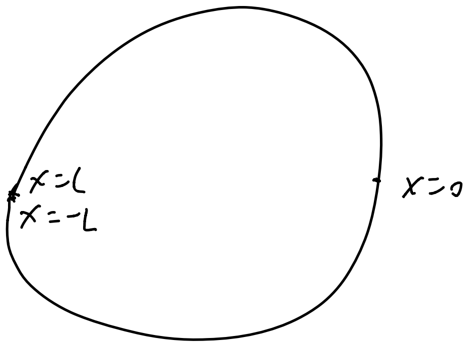
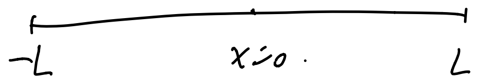
Circular rod:





$u_t = k u_{xx}.$

$$\left.\begin{aligned} u(-L,t) &= u(L,t) \\ u_x(-L,t) &= u_x(L,t). \end{aligned}\right\} \text{BC.}$$

$u(x,0) = f(x).$ (IC)

$$\boxed{\text{Or} \quad (BC) \iff u(x,t) = u(x+2L,t) \quad \text{periodic condition.}}$$

$u(x,t) = \phi(x) \cdot G(t)$

Boundary value problem:

$$\phi'' = -\lambda \phi$$

$$\phi(-L) = \phi(L)$$

$$\phi'(-L) = \phi'(L)$$

$\lambda = 0, \quad \phi(x) = a_0$

$\lambda > 0, \quad \phi(x) = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$

$$\phi'(x) = \sqrt{\lambda}\left(-c_1 \sin\sqrt{\lambda}\,x + c_2 \cos\sqrt{\lambda}\,x\right)$$

$\phi(L) = \phi(-L) \Rightarrow c_1 \cos\sqrt{\lambda}L + c_2 \sin\sqrt{\lambda}L = c_1 \cos(-\sqrt{\lambda}L) + c_2 \sin(-\sqrt{\lambda}L)$.

$c_2 \sin(\sqrt{\lambda}L) = 0$.

$\phi'(L) = \phi'(L) \Rightarrow c_1 \sin(\sqrt{\lambda}L) = 0$.

eigenvalue.

So $\sin(\sqrt{\lambda}L) = 0 \Rightarrow \sqrt{\lambda}L = n\pi$. $\lambda = \left(\frac{n\pi}{L}\right)^2$

$n = 1, 2, \ldots$ eigenfunctions $\phi_n = \cos\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}$.

$$f(x) = a_0 + \sum_{n=1}^{+\infty} a_n \cos\frac{n\pi}{L}x + \sum_{n=1}^{+\infty} b_n \sin\frac{n\pi}{L}x.$$

$$\int_{-L}^{L} \cos\frac{n\pi}{L}x \cos\frac{m\pi}{L}x \, dx = \begin{cases} 0, & m \neq n. \\ 2L, & m = n = 0 \\ L, & m = n \neq 0 \end{cases}$$

$$\int_{-L}^{L} \sin\frac{n\pi}{L}x \sin\frac{m\pi}{L}x \, dx = \begin{cases} 0, & m \neq n \\ L, & m = n > 0. \end{cases}$$

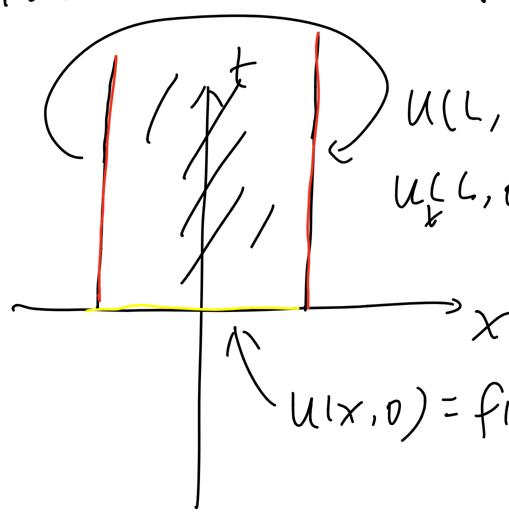
$$\int_{-L}^{L} \sin\frac{n\pi}{L}x \cos\frac{m\pi}{L}x \, dx = 0.$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\, dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\frac{n\pi x}{L}\, dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{n\pi x}{L}\, dx.$$

Idea behind seperation of variables.



$u(L,t) = u(-L,t)$ homogeneous BC.

$u_x(L,t) = u_x(-L,t)$

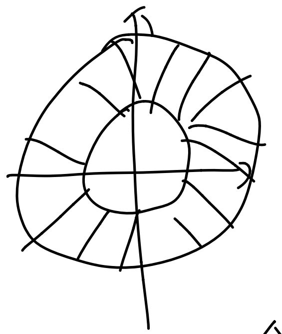
$u(x,0) = f(x)$ inhomogeneous IC

Ignore the inhomogeneous condition.

Use homogeneous BC to solve a Boundary value problem.

Laplace equation in 2D.

Ex: $\Omega = \{ 1 < r < e^2 \} \subset \mathbb{R}^2$.



Solve $\Delta u = 0$. $u(r,\theta)$.

$u(1,\theta) = \sin 2\theta$

$u(e^2,\theta) = 6 + e^4 \sin 2\theta$

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$u(r,\theta) = \phi(\theta)\cdot G(r)$ why? (See the graph at the end)

$$\Delta u = \phi(\theta)\cdot\frac{1}{r}(rG'(r))' + \frac{1}{r^2}\phi''(\theta)G(r) = 0$$

$$\frac{r(rG')'}{G} = -\frac{\phi''(\theta)}{\phi(\theta)} = \lambda$$

$$\begin{cases} \phi''(\theta) = -\lambda\phi \\ \phi(\pi) = \phi(-\pi) \\ \phi'(\pi) = \phi'(-\pi) \end{cases}$$

So $\lambda = \left(\frac{n\pi}{\pi}\right)^2 = n^2$. eigenvalue. $n=0$. $\phi_0 = 1$

$n = 1, 2 \cdots$ $\quad \phi_n = \cos nx$ or $\sin nx$.

$$\frac{r}{G}(rG')' = \lambda = n^2$$

so $\quad r^2 G'' + rG - n^2 G = 0.$

Equidimensional ODE,

$a_0 G, \quad a_1 r G',$
$a_2 r^2 G'', \quad a_3 r^3 G^{(3)}$
$a_4 r^4 G^{(4)} \cdots$

$G(r) = r^p$,

$\Rightarrow \quad p(p-1) + p - n^2 = 0. \Rightarrow p = \pm n$

$n \neq 0, \quad G(r) = C_1 r^n + C_2 r^{-n}$

$n = 0, \quad G(r) = C_1 \quad$ (Missing one solution)

$G(r) = \log r$,

$u(r,\theta)$

$$= A_0 + B_0 \log r + \sum_{n=1}^{+\infty} A_n r^n \cos n\theta + \sum_{n=1}^{+\infty} B_n r^n \sin n\theta$$

$$+ \sum_{n=1}^{+\infty} C_n r^{-n} \cos n\theta + \sum_{n=1}^{+\infty} D_n r^{-n} \sin n\theta$$

$$u(1,\theta)=\sin 2\theta \implies A_0 + B_0\cdot 0 + \sum_{n=1}^{+\infty}(A_n + C_n)\sin n\theta$$
$$+ \sum_{n=1}^{+\infty}(C_n + D_n)\cos n\theta = \sin 2\theta$$

so $A_0 = 0$, $A_n + C_n = 0$,
$C_2 + D_2 = 1$. $C_m + D_m = 0 \quad \forall m \neq 2$

$$u(2,\theta) = 6 + e^4 \sin 2\theta$$

$$\implies B_0 \cdot (\log e^2) + \sum_{n=1}^{+\infty}(A_n e^{2n} + C_n e^{-2n})\sin n\theta$$
$$+ \sum_{n=1}^{+\infty}(C_n e^{2n} + D_n e^{-2n})\cos n\theta$$
$$= 6 + e^4 \sin 2\theta$$

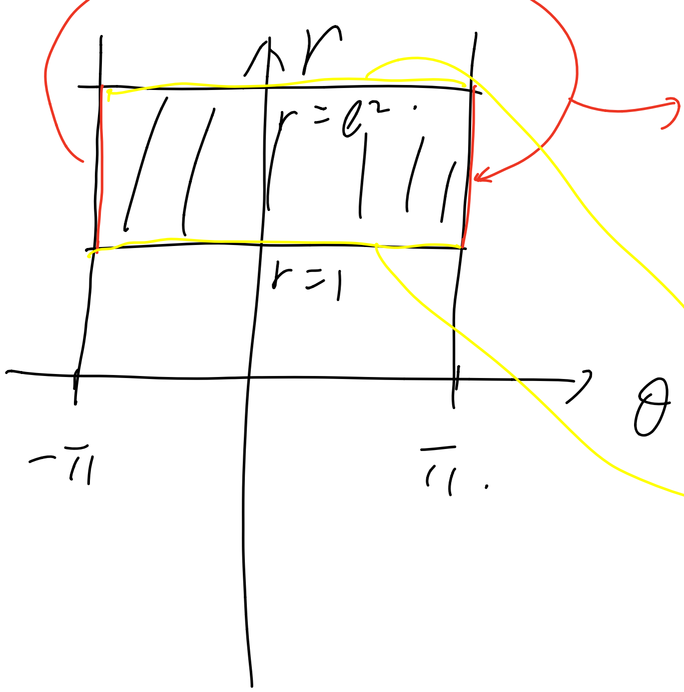
so $B_0 \cdot 2 = 6$, $A_n e^{2n} + C_n e^{-2n} = 0$
$C_2 e^4 + D_2 \cdot e^{-4} = e^4$, $C_m \cdot e^{2m} + D_m e^{-2m} = 0$
$\forall m \neq 2$

so $B_0 = 3$. $A_n = C_n = 0$, $C_2 = 1$. $D_2 = 0$
$C_m = D_m = 0$, $\forall m \neq 2$.

$$u(r,\theta) = 3\log r + r^2 \sin 2\theta$$

why this separation of variables

$$u(r,\theta) = \phi(\theta)\, G(r)$$



$u(r,\pi) = u(r,-\pi)$

$u_\theta(r,\pi) = u_\theta(r,-\pi)$

homogeneous BC

$u(1,\theta) = \sin 2\theta$

$u(e^2,\theta) = 6 + e^4 \sin 2\theta$

inhomogeneous BC

Ignore In hom BC. Use hom. BC to solve Boundary value problem for $\phi(\theta)$

so $u(r,\theta) = \underline{\phi(\theta)}\, G(r)$