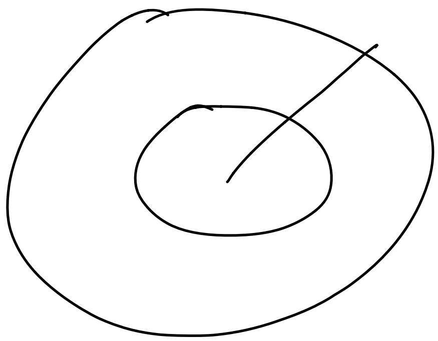
Last time



$$\Delta u = 0$$

$$u(r,\theta)$$

$$= A_0 + B_0 \log r$$

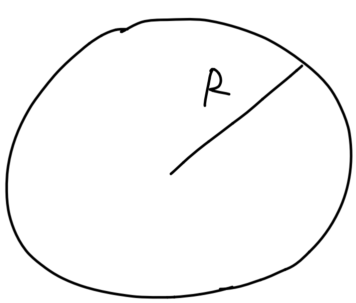
$$+ \sum A_n r^n \cos n\theta$$

$$+ \sum B_n r^n \sin n\theta$$

$$+ \sum C_n r^{-n} \cos n\theta$$

$$+ \sum D_n r^{-n} \sin n\theta$$

How about over a disc



$$u(r,\theta) = \phi(\theta) \cdot G(r)$$

$$|G(0)| < +\infty.$$

so no solutions like $G(r) = \log r$, r^{-n}.

$$u(r,\theta) = a_0 + \sum_{n=1}^{+\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n$$

$$u(r,\theta)\Big|_{r=R} = f(\theta).$$

From othogonality:

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta$$

$$a_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\theta) \cdot \cos n\theta \, d\theta \quad n \geq 1$$

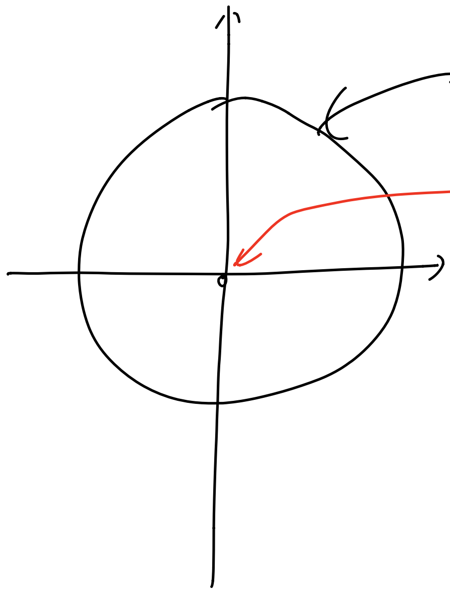
$$b_n = \frac{1}{\pi R^n} \int_0^{2\pi} f(\theta) \cdot \sin n\theta \, d\theta \quad n \geq 1$$

Important property of $\Delta u = 0$

(solutions to Laplace equation are also called harmonic functions)

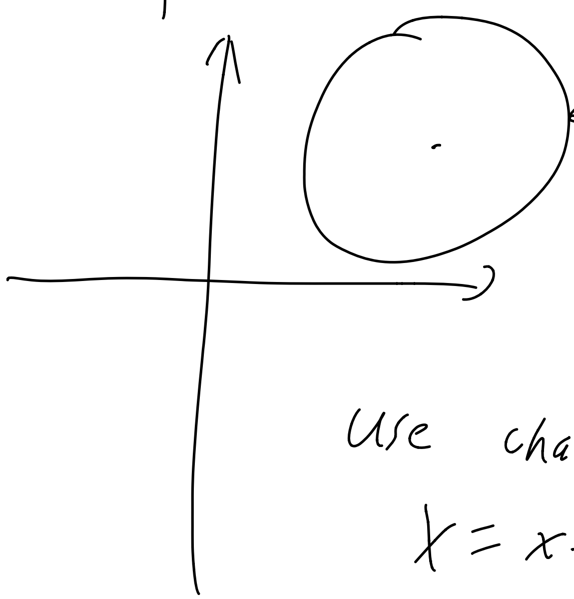
Mean value property:

$$u(0,0) = \frac{1}{2\pi R} \int_{x^2+y^2=R^2} u(x,y)\, ds = \text{average of } u \text{ on circle centered at } (0,0)$$



average. $\bar{u}$

equal to this value at the center.



true for any circle. centered at (a, b).

Use change of coordinates

$$X = x - a$$

$$Y = y - b.$$

$$u_{xx} + u_{yy} = 0 \Rightarrow u_{XX} + u_{YY} = 0.$$

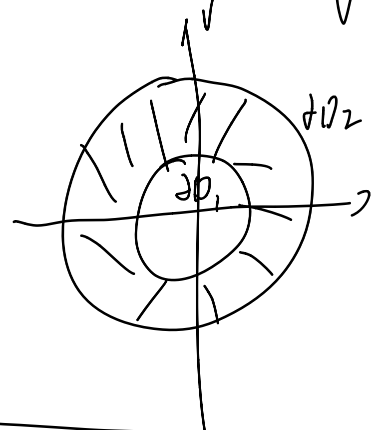
Another proof (Integration by parts) (We will discuss this method later).

$$\int_D f \Delta g = \int_{\partial D} f \frac{\partial g}{\partial \vec{n}} - \int_D \langle \nabla f, \nabla g \rangle$$

$$\int_D f \Delta g - \Delta f\, g = \int_{\partial D} \left(f \frac{\partial g}{\partial \vec{n}} - g \cdot \frac{\partial f}{\partial \vec{n}} \right) \qquad (**)$$

choose $g = \log r$ and.

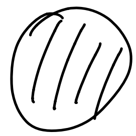
$D =$



we can get $\Delta g = 0$.

$\frac{\partial g}{\partial \vec{n}} = \frac{1}{r}$. $g =$ constant on ∂D_1 and ∂D_2

$$\boxed{\frac{\partial g}{\partial \vec{n}} = \frac{\partial}{\partial r} g.}$$

If $\Delta f = 0$ on  Ω, then $\int_{\partial D} \frac{\partial f}{\partial \vec{n}} = 0$,

$\Rightarrow$ average of f on ∂D_1, and ∂D_2 are the same.

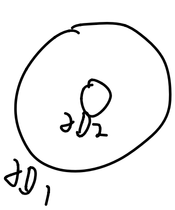
So $$\int_{\partial D_1} g \cdot \frac{\partial f}{\partial \vec{n}} = 0$$

$$\int_{\partial D_2} g \cdot \frac{\partial f}{\partial \vec{n}} = 0.$$

Sub in $(**)$

So $$\int_{\partial D_1} f \cdot \frac{\partial g}{\partial \vec{n}} = \int_{\partial D_1} f \cdot \frac{1}{r} = 2\pi \cdot \text{average of } f \text{ on } \partial D_1$$

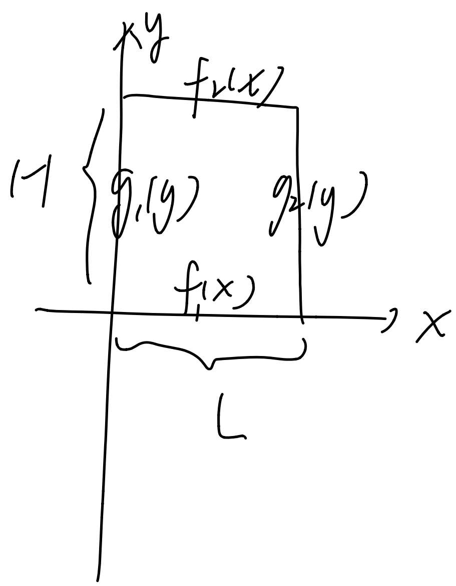
$$= \int_{\partial D_2} f \cdot \frac{\partial g}{\partial \vec{n}} = 2\pi \cdot \text{average of } f \text{ on } \partial D_2.$$



Let ∂D_2 shrink to a point

Then average of f on $\partial D_2 \to f(0,0)$

Solve Laplace equation on a rectangle.



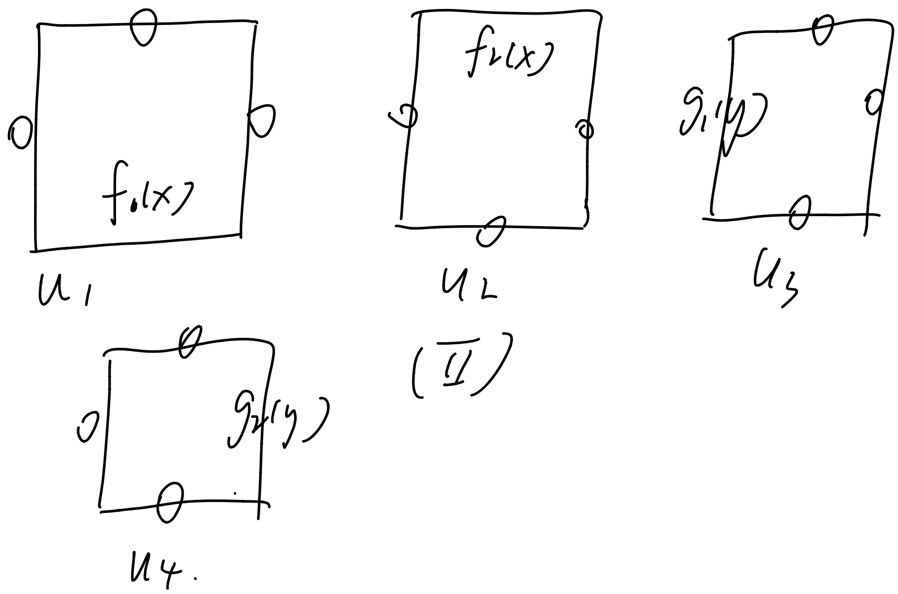
$$u_{xx} + u_{yy} = 0.$$

$$u(x,0) = f_1(x)$$
$$u(x,H) = f_2(x)$$
$$u(0,y) = g_1(y)$$
$$u(L,y) = g_2(y)$$

inhomogeneous

Seperation of variables.

We can only solve it if BCs are homogeneous for one of the two variables and use linear combinations to match the inhomogeneous BCs.



Add u_1, u_2, u_3, u_4 together. we get

u.

$$u_1(x,y) = \phi(x)G(y) \text{ or } \cancel{\phi(y)G(x)}$$

$$\phi'' G + \phi G'' = 0$$

$$\frac{\phi''}{\phi} = -\frac{G''}{G} = -\lambda.$$

$$\phi'' + \lambda \phi = 0, \quad \phi(0) = \phi(L) = 0.$$

So $\lambda_n = (\frac{n\pi}{L})^2$, $\varphi_n(x) = \sin\frac{n\pi x}{L}$.

$n = 1, 2, \cdots$

$$G_n'' = (\frac{n\pi}{L})^2 G_n.$$

$$G(y) = C_1 \cosh\frac{n\pi}{L}y + C_2 \sinh\frac{n\pi}{L}y.$$

$$G(H) = 0.$$

So it is easier to write

$$G(y) = C_1 \cosh\frac{n\pi}{L}(y-H)$$
$$+ C_2 \sinh\frac{n\pi}{L}(y-H)$$

$$G(H) = 0 \implies C_1 = 0$$

$$G(y) = \sinh\frac{n\pi}{L}(y-H)$$

$$u_1(x,y) = \sum_{n=1}^{+\infty} b_n \sinh\frac{n\pi}{L}(y-H) \cdot \sin(\frac{n\pi x}{L})$$

$$b_n = \frac{2}{L \cdot \sinh\frac{n\pi}{L}(-H)} \int_0^L f_2(x) \cdot \sin\frac{n\pi x}{L}\, dx$$

Similarly, we can solve u_2, u_3, u_4

$$u = u_1 + u_2 + u_3 + u_4.$$