Fourier series.

Recall: $u_t = k u_{xx}$.

$u(0,t)=0, \; u(L,t)=0.$

then $u(x,t) = \sum_{n=1}^{+\infty} B_n \sin \frac{n\pi}{L} x \, e^{-k\frac{n^2\pi^2}{L^2}t}.$

at $t=0$ (IC)

$f(x) = \sum_{n=1}^{+\infty} B_n \sin \frac{n\pi x}{L}$ "Sine series"

$x \in [0,L]$

Neumann condition:

$f(x) = \sum_{n=0}^{+\infty} A_n \cos \frac{n\pi}{L} x$ "cosine series".

$x \in [0,L]$

Circular rod:

$f(x) = A_0 + \sum_{n=0}^{+\infty} A_n \cos \frac{n\pi}{L} x + \sum_{n=0}^{+\infty} B_n \sin \frac{n\pi}{L} x$

$x \in [-L, L]$

Full Fourier series

By orthogonality: Full Fourier series:

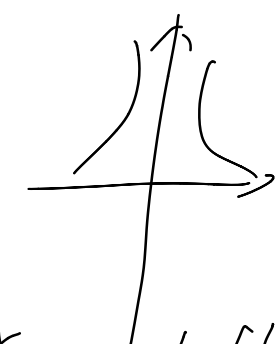
$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\cdot \sin\frac{n\pi x}{L}\,dx \quad n \geq 1.$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \quad n \geq 1$$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx.$$

Question: when does a Fourier series converge to $f(x)$?

Non Ex: $f(x) = \frac{1}{x^2}$, then



$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{L}\frac{1}{x^2}\,dx.$$

$$= +\infty.$$

Defn: $f(x): [a, b) \to \mathbb{R}$ is piecewise smooth

if there exist finitely many

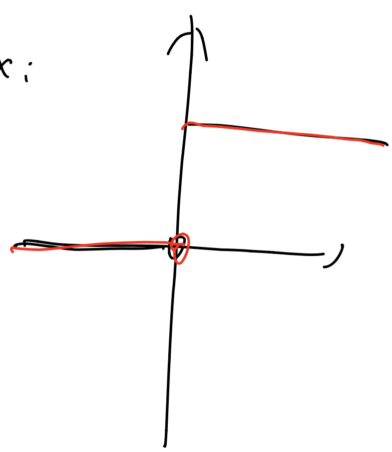
$a = a_0 < a_1 < a_2 < \dots < a_n = b$.

$f\big|_{(a_i, a_{i+1})}$ is smooth.

($f', f'', f''', f^{(n)} \dots$ exist)

and $\lim_{x \to a_i^+} f(x)$, $\lim_{x \to a_i^-} f(x)$ exist

Ex:



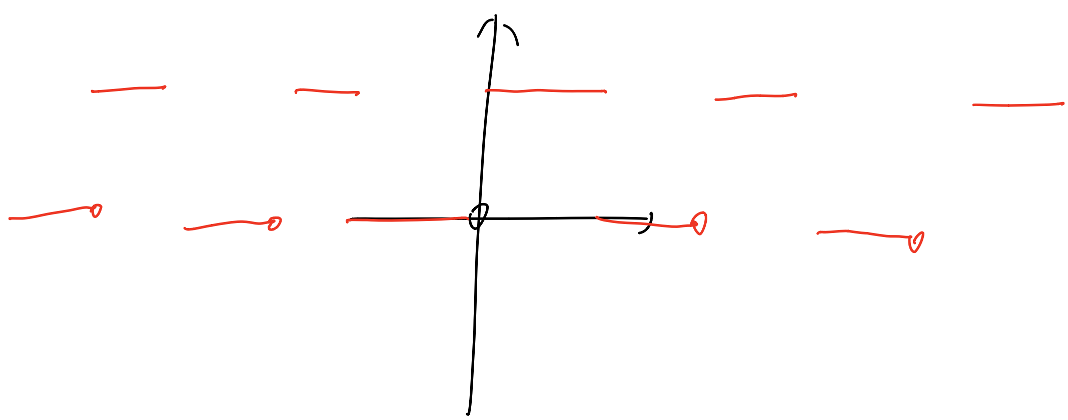
$$f(x) = \begin{cases} 1 & 0 \le x \le L \\ 0 & -L \le x < 0. \end{cases}$$

$\lim_{x \to 0^+} f(x) = 1.$

$\lim_{x \to 0^-} f(x) = 0$

Fourier's Theorem: If $f(x) : [-L, L] \to \mathbb{R}$ is piecewise smooth, then Fourier series of $f(x)$ converges to the periodic extension of $f(x)$ when it is continuous, and to the average of the one sided limits when it is not continuous.

Example: periodic extension.



$L = \pi$, $\quad A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{2}$.

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^{\pi} \cos nx \, dx$$

$$= 0$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{\pi} \sin nx\, dx$$

$$= \frac{1}{n\pi}\left(1 - (-1)^n\right).$$

so $f(x) \sim \frac{1}{2} + \frac{2\sin x}{\pi} + \frac{2\sin 3x}{3\pi} \cdots$

cute fact:

let $x = \frac{\pi}{2}$,

then $1 = \frac{1}{2} + \frac{2}{\pi} - \frac{2}{3\pi} + \frac{2}{5\pi} - \cdots -$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Even and odd functions.

$f(x) = f(-x)$ even x^2 $\cos nx$

$f(-x) = -f(-x)$ odd. x^3 $\sin nx$.

Prop: The full Fourier series of even extension of $f(x)$ is the cosine series of $f(x)$

Prop: The full Fourier series of odd extension of $f(x)$ is the sine series

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Let $\widetilde{f}(x)$ be the even extension.

Then $$A_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\widetilde{f}(x) \sin \frac{n\pi x}{L}}_{\text{odd}}\, dx = 0$$