Math 241 **Name:** ____________________

Fall 2019

Practice exam

9/26/2019

Time Limit: 80 Minutes ID ____________________

"My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this"

Signature ____________________

This exam contains 8 pages (including this cover page) and 6 questions.
Total of points is 120.

- Check your exam to make sure all 8 pages are present.
- You may use writing implements on both sides of a sheet of 5"x7" paper.
- NO CALCULATORS.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
Total:	120	

Boundary value problems:

$$\phi''(x) = -\lambda\phi(x)$$

Boundary conditions	$\phi(0)=0$ $\phi(L)=0$	$\frac{d\phi}{dx}(0)=0$ $\frac{d\phi}{dx}(L)=0$	$\phi(-L)=\phi(L)$ $\frac{d\phi}{dx}(-L)=\frac{d\phi}{dx}(L)$
Eigenvalues λ_n	$\left(\frac{n\pi}{L}\right)^2$ $n=1,\ 2,\ 3,\dots$	$\left(\frac{n\pi}{L}\right)^2$ $n=0,\ 1,\ 2,\ 3,\dots$	$\left(\frac{n\pi}{L}\right)^2$ $n=0,\ 1,\ 2,\ 3,\dots$
Eigenfunctions	$\sin\frac{n\pi x}{L}$	$\cos\frac{n\pi x}{L}$	$\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$
Series	$f(x)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{L}$	$f(x)=\sum_{n=0}^{\infty}A_n\cos\frac{n\pi x}{L}$	$f(x)=\sum_{n=0}^{\infty}a_n\cos\frac{n\pi x}{L}$ $+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}$
Coefficients	$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$	$A_0=\frac{1}{L}\int_0^L f(x)\,dx$ $A_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$	$a_0=\frac{1}{2L}\int_{-L}^L f(x)\,dx$ $a_n=\frac{1}{L}\int_{-L}^L f(x)\cos\frac{n\pi x}{L}\,dx$ $b_n=\frac{1}{L}\int_{-L}^L f(x)\sin\frac{n\pi x}{L}\,dx$

Orthogonality

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = \begin{cases} 0, & n\neq m \\ L/2, & n=m\neq 0 \end{cases}$$

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = \begin{cases} 0, & n\neq m \\ L/2, & n=m\neq 0 \\ L, & n=m=0 \end{cases}$$

$$\int_{-L}^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = \begin{cases} 0, & n\neq m \\ L, & n=m\neq 0 \end{cases}$$

$$\int_{-L}^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = \begin{cases} 0, & n\neq m \\ L, & n=m\neq 0 \\ 2L, & n=m=0 \end{cases}$$

$$\int_{-L}^L \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = 0$$

1. (20 points) Solve the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with $0 \leq x \leq \pi$ and $t \geq 0$ subject to boundary conditions

$$u_x(0,t) = 0, u_x(\pi,t) = 0$$

and initial condition $u(x,0) = 3 + 7\cos 3x$.

2. (20 points) Solve the Laplace equation

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

inside a 60° wedge of radius 2 subject to the boundary conditions

$$u(r,0) = 0, \quad u\left(r, \frac{\pi}{3}\right) = 0, \quad u(2,\theta) = f(\theta)$$

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3. (20 points) Find the equilibrium solution to heat equation $u_t = 2u_{xx} + \sin x$ with $0 \leq x \leq \pi$ subject to boundary conditions

$$u(0,t) = 0, u(\pi,t) = 5.$$

4. (20 points) Solve Laplace equation inside a rectangle $0 \leq x \leq L, 0 \leq y \leq H$ with boundary conditions

$$u(0,y) = 0, \quad u(L,y) = 0, \quad u(x,0) - \frac{\partial u}{\partial y}(x,0) = 0, \quad u(x,H) = f(x).$$

5. (20 points) Consider the Possion equation

$$\Delta u = r^4$$

on the unit disc $D = \{(x, y) | x^2 + y^2 \leq 1\}$.

1. Find one solution $u(r, \theta) = u_0(r)$ only depending on r.
2. Find the solution with boundary condition $u(1, \theta) = \cos(2\theta)$.

6. (20 points) Find the equilibrium solution to 2D heat equation $u_t = \Delta u$ on a unit disc D with insulated boundaries

$$\frac{\partial u}{\partial n} = 0$$

and initial condition $u(x, y, 0) = f(x, y)$.