Math 241 **Name:** ______________________
Fall 2019
Practice exam
10/31/2019
Time Limit: 80 Minutes ID ______________________

"My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this"

Signature ______________________

This exam contains 8 pages (including this cover page) and 6 questions.
Total of points is 120.

- Check your exam to make sure all 8 pages are present.
- You may use writing implements on both sides of a sheet of 5"x7" paper.
- NO CALCULATORS.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
Total:	120	

Boundary value problems:

$$\phi''(x) = -\lambda\phi(x)$$

Boundary conditions	$\phi(0)=0$ $\phi(L)=0$	$\frac{d\phi}{dx}(0)=0$ $\frac{d\phi}{dx}(L)=0$	$\phi(-L)=\phi(L)$ $\frac{d\phi}{dx}(-L)=\frac{d\phi}{dx}(L)$
Eigenvalues λ_n	$\left(\frac{n\pi}{L}\right)^2$ $n=1,\ 2,\ 3,\dots$	$\left(\frac{n\pi}{L}\right)^2$ $n=0,\ 1,\ 2,\ 3,\dots$	$\left(\frac{n\pi}{L}\right)^2$ $n=0,\ 1,\ 2,\ 3,\dots$
Eigenfunctions	$\sin\frac{n\pi x}{L}$	$\cos\frac{n\pi x}{L}$	$\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$
Series	$f(x)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{L}$	$f(x)=\sum_{n=0}^{\infty}A_n\cos\frac{n\pi x}{L}$	$f(x)=\sum_{n=0}^{\infty}a_n\cos\frac{n\pi x}{L}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}$
Coefficients	$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$	$A_0=\frac{1}{L}\int_0^L f(x)\,dx$ $A_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$	$a_0=\frac{1}{2L}\int_{-L}^L f(x)\,dx$ $a_n=\frac{1}{L}\int_{-L}^L f(x)\cos\frac{n\pi x}{L}\,dx$ $b_n=\frac{1}{L}\int_{-L}^L f(x)\sin\frac{n\pi x}{L}\,dx$

Orthogonality

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = \begin{cases} 0, & n\neq m \\ L/2, & n=m\neq 0 \end{cases}$$

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = \begin{cases} 0, & n\neq m \\ L/2, & n=m\neq 0 \\ L, & n=m=0 \end{cases}$$

$$\int_{-L}^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = \begin{cases} 0, & n\neq m \\ L, & n=m\neq 0 \end{cases}$$

$$\int_{-L}^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = \begin{cases} 0, & n\neq m \\ L, & n=m\neq 0 \\ 2L, & n=m=0 \end{cases}$$

$$\int_{-L}^L \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = 0$$

1. (20 points) 1. Compute the Fourier cosine series for the function $f(x) = x^2$ on the interval $[0, \pi]$. Fully simplify your answer. Hint:

$$\int x^2 \cos(nx)\, dx = \frac{2nx\cos(nx) + (-2 + n^2x^2)\sin(nx)}{n^3}$$

2. Does the Fourier cosine series converge to the function f at the point $x = 0$? Justify your answer.

3. Sketch the values of the Fourier cosine series of f on the interval $[-\pi, 2\pi]$, marking any points of discontinuity.

4. Demonstrate from your calculations that

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}.$$

2. (20 points) The displacement of a string $u(x,t)$, $0 \le x \le L$, $t \ge 0$ satisfies the following wave equation

$$u_{tt} = 8u_{xx} - u$$

with boundary conditions $u(0,t) = 0$, $u(L,t) = 0$ and initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. Find the solution $u(x,t)$.

3. (20 points) Consider the boundary value problem

$$x^2\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + x\frac{\mathrm{d}\phi}{\mathrm{d}x} + \lambda\phi = 0, \quad 1 \le x \le e$$

and

$$\phi(1) = \phi(e) = 0.$$

Find the eigenvalues and eigenfunctions.

4. (20 points) 1. Write the following boundary value problem of $\phi(x)$

$$x^2\phi'' + 2x\phi' + \lambda\phi = 0, 1 \leq x \leq 3, \phi(1) = \phi(3) = 0$$

in the standard Sturm–Liouville form.

2. Is this a regular eigenvalue problem? Why?
3. Show that all the eigenvalue $\lambda > 0$.

5. (20 points) Solve the $2D$ wave equation on rectangle $\Omega = [0,4] \times [0,9]$

$$u_{tt} = c^2 \Delta u$$

with boundary conditions $u(x, y, t) = 0$ on the boundary $\partial\Omega$ and initial conditions $u(x, y, 0) = \sin(\pi x)\sin(\pi y)$ and $u_t(x, y, 0) = 0$.

6. (20 points) Consider the heat equation of the temperature $u(x, y, t)$ on a bounded $2D$ region Ω

$$u_t = \Delta u - u, \ (x, y) \in \Omega, \ t \geq 0$$

with boundary condition $u|_{\partial\Omega} = 0$

1. Let

$$E(t) = \iint_{\Omega} (u(x, y, t))^2 \, dxdy.$$

Show that the function $E(t)$ is nonincreasing, in other words

$$\frac{d}{dt} E(t) \leq 0.$$

2. Use the function $E(t)$ to show that the solution with fixed initial condition $u(x, y, 0) = f(x, y)$ is unique.