1. (20 points) Solve the 1D heat equation

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$$

for $0 \le x \le 1$ and $t \ge 0$ subject to boundary conditions

$$u_x(0,t) = 0, \quad u(1,t) = 0$$

and initial condition $u(x,0) = \cos(\frac{\pi}{2}x) + 4\cos(\frac{5\pi}{2}x)$.

$u = \phi(x)\cdot G(t)$

$$\frac{\phi''(x)}{\phi(x)} = \frac{G'(t)}{2G(t)} = -\lambda$$

$\phi'(0) = 0, \quad \phi(1) = 0$

$\lambda > 0 \quad \phi(x) = C_1 \cos(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}x)$

$\phi'(0) = 0 \Rightarrow C_2 = 0.$

$\phi(1) = 0, \quad \sqrt{\lambda} = n\pi + \frac{\pi}{2} \qquad n = 0, 1, 2\ldots$

$\lambda_n = (n\pi + \frac{\pi}{2})^2 \qquad \phi_n(x) = \cos((n\pi + \frac{\pi}{2})x)$

$G_n(t) = e^{-\frac{2(n\pi+\frac{\pi}{2})^2}{\cancel{2}}t}$

$$u(x,t) = \sum_{n=0}^{+\infty} A_n \cos\left((n\pi + \frac{\pi}{2})x\right) \cdot e^{-\frac{(n\pi+\frac{\pi}{2})^2}{2}t}.$$

$u(x,0) = \cos\frac{\pi}{2}x + 4\cos\frac{5\pi}{2}x$

so $A_0 = 1. \quad A_2 = 4. \quad$ other $A_n = 0$

$u(x,t) = \cos\frac{\pi}{2}x \cdot e^{-2(\frac{\pi}{2})^2 t} + 4\cos\frac{5\pi}{2}x\, e^{-2(\frac{5\pi}{2})^2 t}.$

2. (20 points) Consider the 1D heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x.$$

with $0 \leq x \leq 1$ and $t \geq 0$ subject to boundary conditions

$$u_x(0,t) = 1, \quad u_x(1,t) = \beta$$

and initial condition $u(x,0) = f(x)$.

1. For what value of β is there an equilibrium solution?
2. Determine the equilibrium solution.

$u_{xx} = -x.$

$u_x = -\frac{x^2}{2} + C_1$

$u = -\frac{1}{6}x^3 + C_1 x + C_2.$

$u_x(0) = 1 \Rightarrow C_1 = 1.$

$u_x(1) = \beta \Rightarrow -\frac{1}{2} + C_1 = \beta. \Rightarrow \beta = \frac{1}{2}.$

$\int_0^1 -\frac{1}{6}x^3 + x + C_2 \, dx = \int_0^1 f(x)\, dx.$

So $C_2 - \frac{1}{\cancel{18}\ 24} + \frac{1}{2} = \int_0^1 f(x)\, dx$

$C_2 = \cancel{\frac{8}{18}} \cancel{-\frac{4}{9}} + \int_1^1 f(x)\, dx \quad -\frac{11}{24}$

$u(x) = -\frac{1}{6}x^3 + x \cancel{-\frac{4}{9}} + \int_0^1 f(x)\, dx \quad -\frac{11}{24}$

3. (20 points) Solve the Laplace equation

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

in the disk $D = \{(x,y)|x^2+y^2 \le 4\}$ subject to boundary condition

$$\frac{\partial u}{\partial r}(2,\theta) = 32\cos(4\theta) - 8\sin(2\theta).$$

$u(r,\theta) = \phi(\theta)\,G(r)$

$$\frac{\phi''(\theta)}{\phi(\theta)} = -\frac{r(rG_r)_r}{G} = -\lambda.$$

$\phi(-\pi) = \phi(\pi)$
$\phi'(-\pi) = \phi(\pi)$ $\Rightarrow$ $\lambda = 0, 1, 2, \ldots$

$u(r,\theta) = r^4\cos 4\theta - 2r^2\sin 2\theta + A_0$

$\lambda = 0$, $\phi = 1$

$\lambda = n > 0$, $\phi = \sin n\theta$ or $\phi = \cos n\theta$.

Since $|G(0)| < +\infty$, $G(r) = 1.\ r^n \ldots$

so $u(r,\theta) = \sum_{n=0}^{+\infty} A_n r^n \cos n\theta + \sum_{n=1}^{+\infty} B_n r^n \sin n\theta$

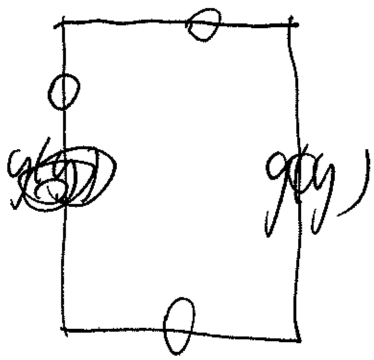
$u_r = \sum_{n=0}^{+\infty} n A_n r^{n-1}\cos n\theta + \sum_{n=1}^{+\infty} n B_n r^{n-1}\sin n\theta$

$u_r(2,\theta) = 32\cos 4\theta - 8\sin 2\theta \Rightarrow A_4 = \frac{32}{4\cdot 2^3} = 1.$

$B_2 = \frac{-8}{2\cdot 2^1} = -2.$ A_0 can be any thing. other coefficient $= 0$

4. (20 points) Solve Laplace equation inside a rectangle $0 \le x \le L, 0 \le y \le H$ with boundary conditions

$$u(0,y) = 0, \quad u(L,y) = g(y), \quad u(x,0) = 0, \quad u(x,H) = 0.$$



$$u(x,y) = \phi(y)\, G(x)$$

$$\frac{\phi''}{\phi} = -\frac{G''}{G} = -\lambda.$$

$$\phi(0) = \phi(H) = 0.$$

$$\lambda = \left(\frac{n\pi}{H}\right)^2, \quad n = 1, 2 \ldots \quad n \ldots$$

$$\phi = \sin \frac{n\pi}{H} y. \qquad G(0) = 0$$

$$G = C_1 \cosh \frac{n\pi}{H}(x) + C_2 \sinh \frac{n\pi}{H}(x)$$

$$G(0) = 0 \Rightarrow C_1 = 0$$

$$u(x,y) = \sum_{n=1}^{+\infty} A_n \cdot \left(\sin \frac{n\pi}{H} y\right) \cdot \sinh \frac{n\pi}{H}(x)$$

$$u(L,y) = g(y)$$

$$\Rightarrow A_n \cdot \sinh \frac{n\pi}{H}(L) = \frac{2}{H} \int_0^H \left(\sin \frac{n\pi}{H} y\right) g(y)\, dy$$

$$A_n = \frac{2}{H \sinh \frac{n\pi}{H}(L)} \cdot \int_0^H \left(\sin \frac{n\pi}{H} y\right) \cdot g(y)\, dy.$$

5. (20 points) Consider the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + tx.$$

with $0 \le x \le 1$ and $t \ge 0$ subject to boundary conditions

$$u_x(0,t) = 0, \quad u_x(1,t) = 0$$

and initial condition $u(x,0) = \frac{1}{2}x^2 - \frac{1}{3}x^3$. Define the heat energy by

$$E(t) = \int_0^1 u(x,t)\,dx.$$

Find $E(t)$.

HW 1, Heat equation
Problem 5

$$\frac{dE}{dt} = \int_0^1 u_t \, dx$$

$$= \int_0^1 u_{xx} + tx \, dx$$

$$= u_x \Big|_0^1 + t\int_0^1 x \, dx$$

$$= t \cdot \frac{1}{2} = \frac{t}{2}.$$

$$E'(t) = \frac{t}{2}, \quad E(t) = \frac{t^2}{4} + C$$

$$E(0) = \int_0^1 \frac{1}{2}x^2 - \frac{1}{3}x^3 \, dx = \frac{1}{6} - \frac{1}{12} = \frac{1}{12}$$

$$E(t) = \frac{t^2}{4} + \frac{1}{12}.$$

6. (20 points) The heat equation

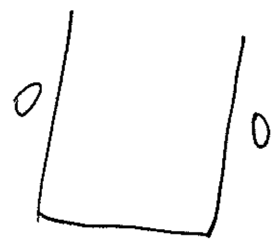
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - u$$

describes the temperature distribution $u(x,t)$ of a 1D rod $0 \le x \le \pi$ suffering some heat loss. Find a solution to the PDE with boundary conditions

$$u_x(0,t) = u_x(\pi,t) = 0$$

and initial condition $u(x,0) = 5 + \cos x + \cos 3x$.

$u(x,t) = \phi(x) G(t)$



$\phi(x) \cdot G'(t) = \phi''(x) G(t) - \phi(x) G(t)$

$$\frac{\phi''}{\phi} = \frac{G'(t)}{G(t)} + 1 = -\lambda.$$

$\phi'' = -\lambda\phi$, $\phi'(0) = 0$, $\phi'(\pi) = 0$.

$\lambda = 0$, $\phi = 1$

$\lambda = n^2$, $\phi = \cos nx$.

$G'(t) = (-\lambda - 1) G(t) \Rightarrow G(t) = e^{(-\lambda-1)t}$.

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos nx \cdot e^{-(n^2+1)t}$$

$u(x,0) = 5 + \cos x + \cos 3x$. $A_0 = 5$. $A_1 = 1$ $A_3 = 1$.

$u(x,t) = 5e^{-t} + \cos x \cdot e^{-2t} + \cos 3x \cdot e^{-10t}$.