Math 241 **Name:** ____________________

Fall 2019

Midterm 2

10/31/2019

Time Limit: 80 Minutes ID ____________________

"My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this"

Signature ____________________

This exam contains 10 pages (including this cover page) and 6 questions.
Total of points is 120.

- Check your exam to make sure all 10 pages are present.
- You may use writing implements on both sides of a sheet of 5"x7" paper.
- NO CALCULATORS.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
Total:	120	

Boundary value problems:

$$\phi''(x) = -\lambda\phi(x)$$

Boundary conditions	$\phi(0)=0$ $\phi(L)=0$	$\frac{d\phi}{dx}(0)=0$ $\frac{d\phi}{dx}(L)=0$	$\phi(-L)=\phi(L)$ $\frac{d\phi}{dx}(-L)=\frac{d\phi}{dx}(L)$
Eigenvalues λ_n	$\left(\frac{n\pi}{L}\right)^2$ $n=1,\ 2,\ 3,\ldots$	$\left(\frac{n\pi}{L}\right)^2$ $n=0,\ 1,\ 2,\ 3,\ldots$	$\left(\frac{n\pi}{L}\right)^2$ $n=0,\ 1,\ 2,\ 3,\ldots$
Eigenfunctions	$\sin\frac{n\pi x}{L}$	$\cos\frac{n\pi x}{L}$	$\sin\frac{n\pi x}{L}$ and $\cos\frac{n\pi x}{L}$
Series	$f(x)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{L}$	$f(x)=\sum_{n=0}^{\infty}A_n\cos\frac{n\pi x}{L}$	$f(x)=\sum_{n=0}^{\infty}a_n\cos\frac{n\pi x}{L}$ $+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}$
Coefficients	$B_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$	$A_0=\frac{1}{L}\int_0^L f(x)\,dx$ $A_n=\frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$	$a_0=\frac{1}{2L}\int_{-L}^L f(x)\,dx$ $a_n=\frac{1}{L}\int_{-L}^L f(x)\cos\frac{n\pi x}{L}\,dx$ $b_n=\frac{1}{L}\int_{-L}^L f(x)\sin\frac{n\pi x}{L}\,dx$

Orthogonality

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = \begin{cases} 0, & n\neq m \\ L/2, & n=m\neq 0 \end{cases}$$

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = \begin{cases} 0, & n\neq m \\ L/2, & n=m\neq 0 \\ L, & n=m=0 \end{cases}$$

$$\int_{-L}^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} = \begin{cases} 0, & n\neq m \\ L, & n=m\neq 0 \end{cases}$$

$$\int_{-L}^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = \begin{cases} 0, & n\neq m \\ L, & n=m\neq 0 \\ 2L, & n=m=0 \end{cases}$$

$$\int_{-L}^L \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = 0$$

1. (20 points) 1. Compute the Fourier sine series for the function $f(x) = x$ on the interval $[0, \pi]$. Fully simplify your answer. Hint:

$$\int x \sin(nx)\, dx = \frac{-nx\cos(nx) + \sin(nx)}{n^2}$$

2. Does the Fourier sine series converge to the function f at the point $x = 0$? Justify your answer.

3. Sketch the values of the Fourier sine series of f on the interval $[-\pi, 2\pi]$, marking any points of discontinuity.

4. Demonstrate from your calculations that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

Hint: use $f(\frac{\pi}{2})$.

1. $f(x) \sim \sum a_n \sin nx$.

$$a_n = \frac{2}{\pi}\int_0^\pi x \sin nx$$

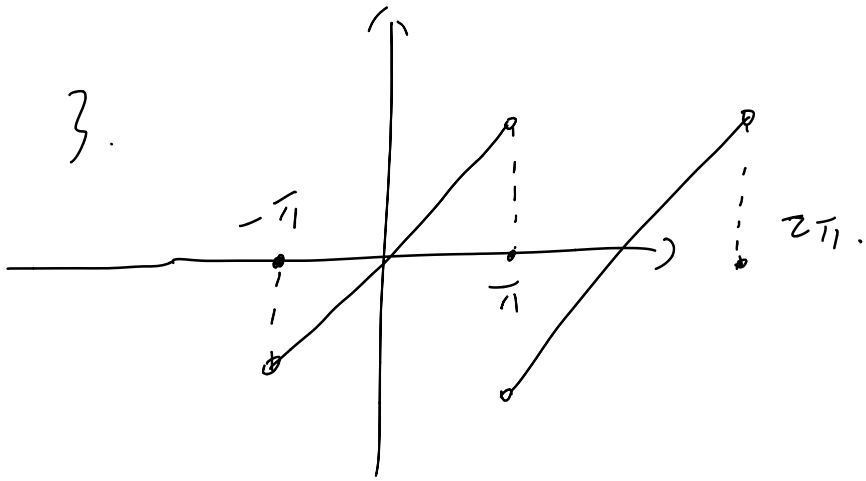
$$= \frac{2}{\pi} \left. \frac{-nx\cos nx + \sin nx}{n^2} \right|_0^\pi$$

$$= \frac{2}{\pi} \frac{-n\pi}{n^2} \cos n\pi$$

$$= -\frac{2}{n}(-1)^n = \frac{(-1)^{n-1}\cdot 2}{n}.$$

2. Yes. the odd extension of x on $[-\pi, \pi]$ is $f(x) = x$

3.



4. $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$.

$$\frac{\pi}{2} = \sum \frac{(-1)^{n-1} 2}{n} \sin n \cdot \frac{\pi}{2}$$

$$= 2 - \frac{2}{3} + \frac{2}{5} - \frac{2}{7} + \cdots$$

so $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$

2. (20 points) Solve $1D$ wave equation

$$u_{tt} = u_{xx} - u, \quad 0 \le x \le L,\ t \ge 0$$

with boundary conditions $u_x(0,t) = 0$ and $u_x(L,t) = 0$ and initial conditions $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$.

$$u(x,t) = \phi(x) \cdot G(t)$$

$$G''\phi = \phi'' G - \phi \cdot G.$$

$$\frac{G''}{G} + 1 = \frac{\phi''}{\phi} = -\lambda$$

$$\begin{cases} \phi'' = -\lambda \phi \\ \phi'(0) = \phi'(L) = 0. \end{cases}$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \phi_n = \cos \frac{n\pi x}{L}. \quad n = 0, 1, \ldots$$

$$G'' = (-\lambda - 1) G \Rightarrow G = \begin{cases} \cos \sqrt{\lambda+1}\, t \\ \sin \sqrt{\lambda+1}\, t. \end{cases}$$

$$u(x,t) = \sum a_n \cos \frac{n\pi x}{L} \cos \sqrt{\left(\frac{n\pi}{L}\right)^2 + 1}\, t$$

$$+ \sum b_n \cos \frac{n\pi x}{L} \sin \sqrt{\left(\frac{n\pi}{L}\right)^2 + 1}\, t.$$

$$u(x,0) = \sum a_n \cos \frac{n\pi x}{L}$$

$$u_t(x,0) = \sum b_n \sqrt{\left(\frac{n\pi}{L}\right)^2 + 1} \cos \frac{n\pi x}{L}.$$

$$a_n = \begin{cases} \frac{1}{L} \int_0^L f(x)\,dx & n=0. \\ \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\,dx & n \geq 1. \end{cases}$$

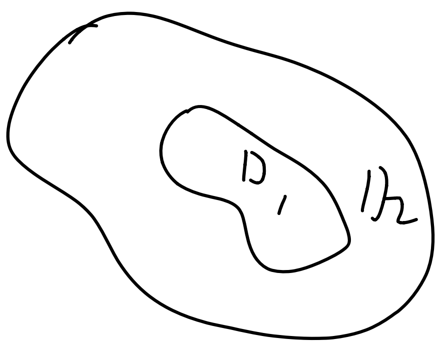
$$b_n = \begin{cases} \frac{1}{L} \int_0^L g(x)\,dx & n=0. \\ \frac{2}{L\sqrt{\left(\frac{n\pi}{L}\right)^2+1}} \int_0^L g(x) \cdot \cos \frac{n\pi x}{L}\,dx & n \geq 1. \end{cases}$$

3. (20 points) Let Ω be a bounded region in $2D$ plane, which contains the square $[0, L] \times [0, L]$ and is contained in the rectangle $[-L, 2L] \times [-L, 3L]$. Let λ_1 be the lowest eigenvalue of Laplacian

$$\Delta\phi(x, y) + \lambda\phi(x, y) = 0$$

with Dirichlet boundary condition $\phi|_{\partial\Omega} = 0$. Find m_1 and m_2 such that $m_1 \le \lambda_1 \le m_2$.

We use the fact that



D_1 is contained in D_2

then $\lambda_1(D_1) \ge \lambda_1(D_2)$

so $\lambda_1([-L, 2L] \times [-L, 3L])$

$\le \lambda_1(\Omega) \le \lambda_1([0, L] \times [0, L])$

$m_1 = \left(\frac{\pi}{3L}\right)^2 + \left(\frac{\pi}{4L}\right)^2 = \frac{\pi^2}{L^2}\left(\frac{1}{9} + \frac{1}{16}\right)$

$m_2 = \left(\frac{\pi}{L}\right)^2 + \left(\frac{\pi}{L}\right)^2 = \frac{2\pi^2}{L^2}.$

4. (20 points) 1. Write the following boundary value problem in the standard Sturm-Louville form

$$x^2\phi'' + 3x\phi' - 5x\phi + \lambda\phi = 0,\ 1 \le x \le 3,\ \phi'(1) = 0,\ \phi(3) = 0$$

2. Is this a regular Sturm-Louville problem? Justify your answer.
3. Show that all the eigenvalues $\lambda \geq 5$. Is $\lambda = 5$ an eigenvalue?

1. $\phi'' + \frac{3}{x}\phi' - \frac{5}{x}\phi + \frac{\lambda}{x^2}\phi = 0.$

Integration factor $e^{\int \frac{3}{x}} = e^{3\log x}$

$= x^3.$

So $x^3\phi'' + 3x^2\phi' - 5x^2\phi + \lambda x\phi = 0$

$(x^3\phi')' - 5x^2\phi + \lambda x\phi = 0.$

2. Yes, $x^3 > 0$, $x > 0$ on $[1,3]$

3. $\lambda = \dfrac{\int x^3(\phi')^2 dx + \int \phi^2 \cdot 5x^2 dx}{\int \phi^2 \cdot x\, dx}$

$x \geq 1 \Rightarrow 5x^2 \geq 5x \Rightarrow \int \phi^2 \cdot 5x^2 dx$

$\geq \int \phi^2 \cdot 5x\, dx$

So $\lambda = \dfrac{\int x^3 (\phi')^2 dx}{\int \phi^2 \cdot x \, dx} + \dfrac{\int \phi^2 \cdot 5x^2 dx}{\int \phi^2 x \, dx}$

$\geq \dfrac{\int x^3 (\phi')^2 dx}{\int \phi^2 \cdot x \, dx} + 5 \geq 5$

If $\lambda = 5$, then $\phi' = 0$.

$\phi =$ constant. on the other hand

$\phi(3) = 0 \Rightarrow \phi = 0$.

So $\lambda = 5$ is not an eigenvalue.

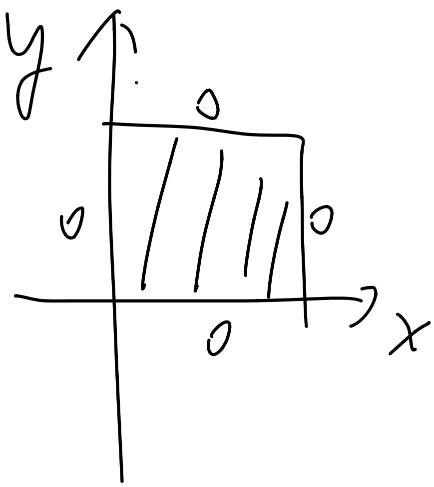
5. (20 points) Solve the 3D Laplace equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

on a cube $[0,L]\times[0,L]\times[0,L]$ assuming

$$u(0,y,z)=0, u(L,y,z)=0, u(x,0,z)=0, u(x,L,z)=0,$$

$$u(x,y,0)=0, u(x,y,L)=f(x,y)$$



$u(x,y,z)=\phi(x,y)\cdot G(z)$

PDE $\Rightarrow \Delta\phi\, G+\phi\, G''=0$

$$\frac{\Delta\phi}{\phi}=-\frac{G''}{G}=-\lambda$$

$\Delta\phi=-\lambda\phi,\quad \lambda_{mn}=\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{L}\right)^2$

$\phi_{mn}=\sin\frac{n\pi x}{L}\sin\frac{m\pi}{L}y$

$G=C_1\sinh\sqrt{\lambda}\,z+C_2\cosh\sqrt{\lambda}\,z$

$G(0)=0 \Rightarrow C_2=0.$

So $u(x,y,z)=\sum a_{mn}\sin\frac{n\pi}{L}x\sin\frac{m\pi}{L}y\cdot\sinh\sqrt{\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{L}\right)^2}\,z.$

$$a_{mn} = \frac{4}{L^2 \sinh\left(\sqrt{\left(\frac{n\pi}{L}\right)^2+\left(\frac{m\pi}{L}\right)^2}\, L\right)}$$

$$\cdot \int_0^L \int_0^L f(x,y) \cdot \sin \frac{n\pi}{L} x \sin \frac{m\pi}{L} \, dx\, dy$$

6. (20 points) Consider the $2D$ wave equation on a bounded region Ω

$$u_{tt} = \Delta u - 3u$$

with Dirichlet boundary condition that $u(x, y, t) = 0$ on the boundary of Ω.

1. The energy function $E(t)$ is defined by

$$E(t) = \frac{1}{2}\iint_\Omega (u_t)^2 + |\nabla u|^2 + 3u^2\,dxdy.$$

   Show that $E(t)$ is a constant.
   Hint: integration by parts

$$\iint_\Omega f\Delta g = \int_{\partial\Omega} f\frac{\partial g}{\partial n} - \iint_\Omega \langle \nabla f, \nabla g\rangle$$

   Here $\frac{\partial g}{\partial n}$ is the directional derivative of g in outward normal direction n.

2. Use this fact to show the uniqueness of solution with fixed initial conditions $u(x, y, 0) = f(x, y)$ and $u_t(x, y, 0) = g(x, y)$.

$$|\nabla u|^2 = (u_x)^2 + (u_y)^2$$

1. $$\frac{d}{dt}E(t) = \iint_\Omega u_t \cdot u_{tt} + u_{xt}u_x + u_{yt}u_y + 3u_t u \, dxdy$$

$$= \iint_\Omega u_t \cdot (\Delta u \cancel{-3u}) + u_{xt}u_x + u_{yt}u_y \cancel{+ 3u_t u} \, dxdy$$

$$= \int_{\partial\Omega} u_t \frac{\partial u}{\partial \vec{n}} - \iint_\Omega \langle \nabla u_t, \nabla u\rangle + \iint_\Omega u_{xt}u_x + u_{yt}u_y$$

$$= 0 - \iint u_{xt}u_x + u_{yt}u_y + \iint u_{xt}u_x + u_{yt}u_y$$

$$= 0.$$

2. Assume u_1, u_2 are two solutions with IC $u(x,y,0)=f(x,y)$

then $v = u_1 - u_2$ is also a solution with IC

$v(x,y,0)=0$.

$E(0)=0$, $E(t)=E(0)=0$

so $v(x,y,t)=0$

$\Rightarrow u_1(x,y,t)=u_2(x,y,t)$

Draft 1:

If you use this page and want it looked at, then you must indicate so on the page with the original problem on it. Make sure you label your work with the corresponding problem number.

Draft 2:

If you use this page and want it looked at, then you must indicate so on the page with the original problem on it. Make sure you label your work with the corresponding problem number.