Math 371 Homework#12

We will discuss these examples on Tuesday's class

1. Find the degree of the splitting field K of the following polynomials over $\mathbb{Q}$.
   (a) $x^3 - 2$
   (b) $x^4 - 1$
   (c) $x^4 + 1$
2. Determine the Galois group of the field extensions in question 1 and use the groups to find the intermediate extensions L in $\mathbb{Q} \subset L \subset K$.

1. a) $x^3-2 = (x-\sqrt[3]{2})(x-\sqrt[3]{2}w)(x-\sqrt[3]{2}w^2)$

$$w = e^{\frac{2\pi i}{3}}, \quad w^2+w+1=0.$$

So $K = \mathbb{Q}(\sqrt[3]{2}, w)$

From the extension diagram

$$\begin{tikzcd} & K=\mathbb{Q}(\sqrt[3]{2},w) \arrow[dl, dash, "\geq 3"'] \arrow[dr, dash, "\leq 2"] \arrow[dd, dash] & \\ \mathbb{Q}(\sqrt[3]{2}) \arrow[dr, dash, "3"'] & & \mathbb{Q}(w) \arrow[dl, dash, "2"] \\ & \mathbb{Q} & \end{tikzcd}$$

$3 \mid [K:\mathbb{Q}]$. $2 \mid [K:\mathbb{Q}]$. $[K:\mathbb{Q}] \leq 6$.

So $[K:\mathbb{Q}] = 6$.

b) $x^4 - 1 = (x^2+1)(x^2-1)$

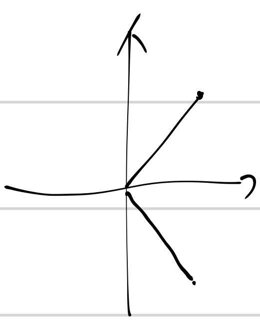
$= (x+i)(x-i)(x+1)(x-1)$

so $K = \mathbb{Q}(i)$

$[K : \mathbb{Q}] = 2$.

c). $x^4 + 1 = (x^2+i)(x-i)$

$= \left(x + \frac{\sqrt{2}+\sqrt{2}i}{2}\right)\left(x - \frac{\sqrt{2}+\sqrt{2}i}{2}\right)$



$\left(x - \frac{\sqrt{2}-\sqrt{2}i}{2}\right)\left(x + \frac{\sqrt{2}-\sqrt{2}i}{2}\right)$

so $K = \mathbb{Q}\left(\frac{\sqrt{2}+\sqrt{2}i}{2}, \frac{-\sqrt{2}-\sqrt{2}i}{2}, \frac{\sqrt{2}-\sqrt{2}i}{2}, \frac{-\sqrt{2}+\sqrt{2}i}{2}\right)$

$= \mathbb{Q}(\sqrt{2}, i)$

since $i^2 + 1 = 0$.

so $[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}(\sqrt{2})] \le 2$.

on the other hand

$i \notin \mathbb{Q}(\sqrt{2})$

so $[\mathbb{Q}(\sqrt{2}, i) : \mathbb{Q}(\sqrt{2})] = 2$

and $[K : \mathbb{Q}] = 4$.

2. a). See Example 3 in 04/30's notes. $G(K/\mathbb{Q}) \cong S_3$.

and we have 6 intermediate field extensions.

b). See Example 2 in 04/30's notes. It's similar to the case $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$

$G(K/\mathbb{Q}) \cong C_2 \times C_2$

we have 5 intermediate field extensions

$$\begin{tikzcd}[arrows=dash]
 & K = \mathbb{Q}(\sqrt{2}, i) \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathbb{Q}(i) \arrow[dr] & \mathbb{Q}(\sqrt{2}) \arrow[d] & \mathbb{Q}(\sqrt{-2}) \arrow[dl] \\
 & \mathbb{Q} &
\end{tikzcd}$$

c). See Example 1 on 04/30's notes.