HW2.

1. ① closure.

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \cdot \begin{bmatrix} a' & b' \\ -b' & a' \end{bmatrix} = \begin{bmatrix} aa'-bb' & ab'+ba' \\ -ba'-ab' & aa'-bb' \end{bmatrix}$$

② Inverse.

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}^{-1} = \frac{1}{a^2+b^2} \cdot \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

③ Identity $\begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \in G$

Map $\rho: G \to \mathbb{C}^\times$

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mapsto a+b\sqrt{-1}$$

$$\rho\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \cdot \begin{bmatrix} a' & b' \\ -b' & a' \end{bmatrix}\right) = (aa'-bb') + (ab'+ba')\sqrt{-1}.$$

$$\rho\left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right) \cdot \rho\begin{pmatrix} a' & b' \\ -b' & a' \end{pmatrix} = (a+b\sqrt{-1})(a'+b'\sqrt{-1})$$

$$= aa'-bb' + (ab'+ba')\sqrt{-1}$$

so ρ is a group homomorphism.

2. "if". if G is abelian. $\forall a, b \in G$.

then $(ab)^{-1} = b^{-1}a^{-1} = a^{-1} \cdot b^{-1}$.

So $a \mapsto a^{-1}$ is a group homomorphism.

"only if". $\forall a, b \in G$. $(ab)^{-1} = a^{-1}b^{-1}$

so $(ab)^{-1} = a^{-1}b^{-1} = (ba)^{-1}$.

$\Rightarrow$ $ab = ba$

3. a). ① closure.

$\forall h_1, h_2 \in H$. $k_1, k_2 \in K$.

$(h_1 k_1)(h_2 k_2) = h_1 (k_1 h_2 k_1^{-1}) k_1 k_2$.

Because H is normal,

we have $k_1 h_2 k_1^{-1} \in H$

so $h_1 k_1 h_2 k_1^{-1} \in H$, $k_1 k_2 \in H$

② Inverse: $\forall h \in H$, $k \in K$,

$(hk)^{-1} = k^{-1}h^{-1} = (k^{-1}h^{-1}k)k^{-1}$

H is normal $\Rightarrow$ $k^{-1}h^{-1}k \in H$.

so $(hk)^{-1} \in HK$

③ Identity $1 \cdot 1 = 1 \in HK$

b). $\forall\, h \in H \cap K,\ k \in K,$

$khk^{-1} \in H$ because H is normal in G.

$k, h \in K$, so $khk^{-1} \in K$,

so $khk^{-1} \in H \cap K$

c). K is a subgroup of HK.

$\rho: K \to HK \to HK/H$.

$k \mapsto k \mapsto k \cdot H$.

Restrict the canonical homomorphism $HK \to HK/H$ to K, we get

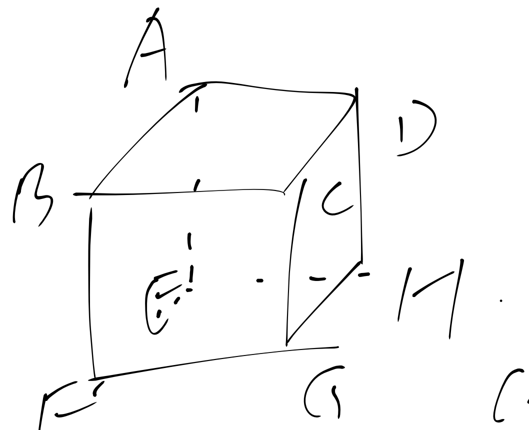
$\rho: K \to HK/H$.

$k \in \ker\rho$ iff $kH = H$. so

$\ker \rho = K \cap H$.

Apply first isomorphism theorem.

We get $K / H \cap K \cong HK / H$

4. 

G = rotational symmetry of the cube ABCDEFGH.

Consider G acts on 6 faces.

Stabilizer of face s = ABCD is the rotational symmetry of square ABCD.

so $|G_s| = 4$.

On the other hand, G action on 6 faces is transitive,

so $|G| = 4 \cdot 6 = 24$.

5. Apply counting formulas to conjugation operation of G on G.

From definition of C_i,

C_i is the stabilizer of x_i under conjugation.

So $|G| = |O_i| \cdot |C_i|$.

on the other hand, $|G| = |O_1| + |O_2| \cdots + |O_k|$.

$$\frac{1}{n_i} = \frac{1}{|C_i|} = \frac{|O_i|}{|G|}.$$

So $$\frac{1}{n_1} + \cdots \frac{1}{n_k} = \frac{|O_1|}{|G|} + \cdots \frac{|O_k|}{|G|} = 1.$$

6. Pf: $$(PQ) * A = (PQ)A(PQ)^t$$
$$= P(QAQ^t)P^t$$
$$= P * (Q * A)$$

7. It's equivalent to describle the group homomophism $\rho: S_3 \to S_4$.

Let $x = (1\,2\,3)$ $y = (12)$

then $x^3 = 1$, $y^2 = 1$.

$$yxy^{-1} = x^2.$$

so $(\rho(x))^3 = 1$. $(\rho(y))^2 = 1$.

case 1). $\rho(x) = 1$. then $\rho: S_3 \to S_3/\langle x \rangle \to S_4$.

1.1) $\rho(y) = 1$. ρ is trivial homomophism.

$\rho(S_3) = \{1\}$.

1.2) $\rho(y) \neq 1$ is an element with order 2 in S_4.

After reindexing the 4 elements.

we can assume $\rho(y) = (12)$ ... case 1.2.1)

or $\rho(y) = (12)(34)$ ..case 1.2.2)

Case 2). $\rho(x) \neq 1$, is an element with order 3.

then after reindexing the four elements.

we have $\rho(x) = (123)$

then $\rho(x^2) = (\rho(x))^2 = (132)$.

$$\rho(yxy^{-1}) = \rho(y)\rho(x)\rho(y)^{-1}$$

$$\Rightarrow \rho(y)\cdot(123)\,\rho(y)^{-1} = (132)$$

On the other hand.

$$\rho(y)(123)\rho(y)^{-1} = (\rho(y)\cdot(1)\ \rho(y)\cdot(2)\ \rho(y)\cdot(3))$$

so $(\rho(y)(1)\ \ \rho(y)(2)\ \ \rho(y)(3))$

$= (1\ 3\ 2)$

There are three possibilities.

2a) $\rho(y)(1) = 1$, $\rho(y)(2) = 3$,

$\rho(y)\cdot(3) = 2$. $\rho(y) = (23)$.

2b) $\rho(y)(1)=3$, $\rho(y)(2)=2$

$\rho(y)(3)=1$. $\rho(y)=(13)$

2c) $\rho(y)(1)=2$, $\rho(y)(2)=1$

$\rho(y)(3)=3$. $\rho(y)=(12)$

After reindexing 1, 2, 3 by (123) or (132), these three cases 2a), 2b), 2c) are the same.

So up to reindexing the four elements, we get 4 different actions of S_3 on the set of four elements.