Solutions to HW3

1. Pf: "If": $Hg = gH$

$\Rightarrow \forall g \in G, h \in H.$

$gh \in Hg$, so $\exists h' \in H$,

s.t. $gh = h'g$.

So $ghg^{-1} = h' \in H$.

so H is normal subgroup

"only if". $\forall g \in G, h \in H$.

$gh = (ghg^{-1})g$.

H is normal $\Rightarrow ghg^{-1} \in H$,

so $(ghg^{-1})g \in Hg$.

So $gH \subset Hg$. Similarly $Hg \subset gH$ $\Rightarrow$ $Hg = gH$

2. $\forall g \notin H$. $G = H \sqcup gH$

because $|G/H| = 2$.

and $G = H \sqcup Hg$.

so $gH = G - H = Hg$

From problem 1. H is normal subgroup.

3. $\forall g \in G$, $h \in H$. $ghg^{-1} \in H$.

so $\forall h \in H$. $\{ghg^{-1} \mid g \in G\} \subset H$.

so $H = \bigcup_{h \in H} \{ghg^{-1} \mid g \in G\}$.

is the union of some conjugacy classes in G.

4. Let $S = \{ i \in \{1 \ldots n\} \mid x(i) \neq i \}$.

Use induction on $|S|$.

If $|S| = 0$, then $x = 1$

If x is a product of transpositions for $|S| \leq k-1$.

then for any x with $|S| = k$.

Let $i_0 \in S$, $x(i_0) \neq i_0$.

choose transposition $y = (i_0, x(i_0))$

Consider yx.

$x(i_0) \neq i_0 \Rightarrow x(x(i_0)) \neq x(i_0)$.

$x(i_0) \in S$.

If $j \notin S$, then $j \neq i_0$ or $x(i_0)$.

so $yx(j) = y(j) = j$.

If $j = i_0$, then $yx(i_0) = i_0$,

so $\{i \in \{1 \dots n\} \mid yx(i) \neq i\} \subset S - \{i_0\}$.

From the induction assumption,

$yx = y_1 \cdots y_k$ with y_i are transpositions.

so $x = y y_1 \cdots y_k$ is a product of transpositions.

5. a). Partitions of 4.

$4 = 1+1+1+1$

$= 1+3$

$= 2+2$.

$= 1+1+2$.

$= 4$.

There're 5 conjugacy classes.

b) $1+1+1+1 \Rightarrow \{1\}$. 1 element

$1+3 \Rightarrow \{(123), (134), (124), (234)$
$(132), (143), (142), (243)\}$.
8 elements

$2+2 \Rightarrow \{(12)(34), (14)(23), (13)(24)\}$
3 elements

$1+1+2 \Rightarrow \{(12), (34), (13), (24),$
$(14), (23)\}$.
6 elements.

$4 \Rightarrow \{(1234), (1324), (1423),$
$(1243), (1342), (1432)\}$.
6 elements.

c). $|S_4| = 24$, has divisors

$1, 2, 4, 8, 3, 6, 12, 24$

so $1,\ 1+3,\ 1+3+8,\ 1+3+6+6+8$

are all the union of conjugacy classes

such that the order divides 24,

check:

d). $\{1\}$.

$\{1, (12)(34), (13)(24), (14)(23)\}$.

$A_4 \subset S_4$,

S_4

are all normal subgroups of G.

6. a). Proof: C_n is a finite subgroup of $SO(2)$

so $G \subset C_n$ is a subgroup of $SO(2)$.

so $G \cong C_m$ for some m.

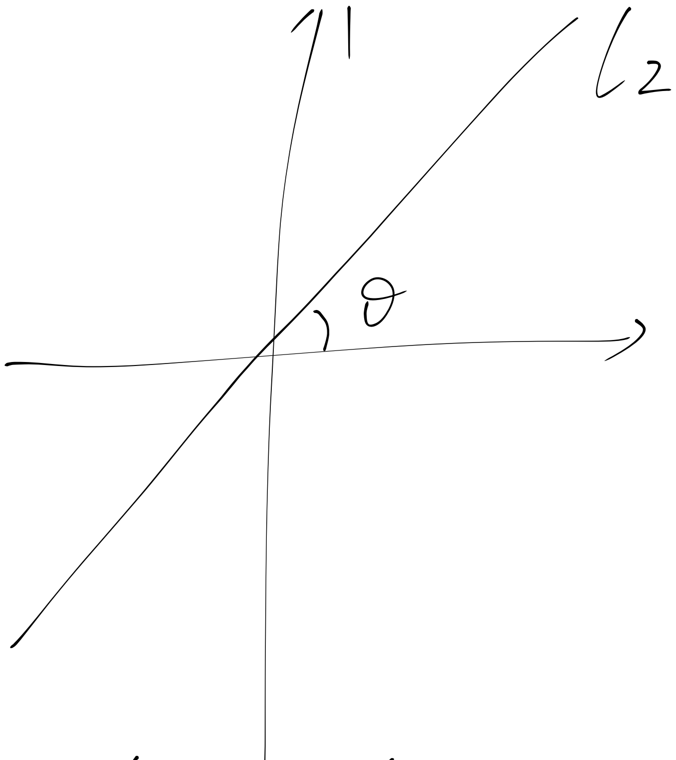
b) pf: D_n is a finite subgroup of $O(2)$.

so $G \subset D_n$ is also a subgroup of $O(2)$, hence is a cyclic group or a dihedral group.

7. Assume l_1 is x-axis.

then $y_1 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$

If l_2 is



then $y_2 = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$.

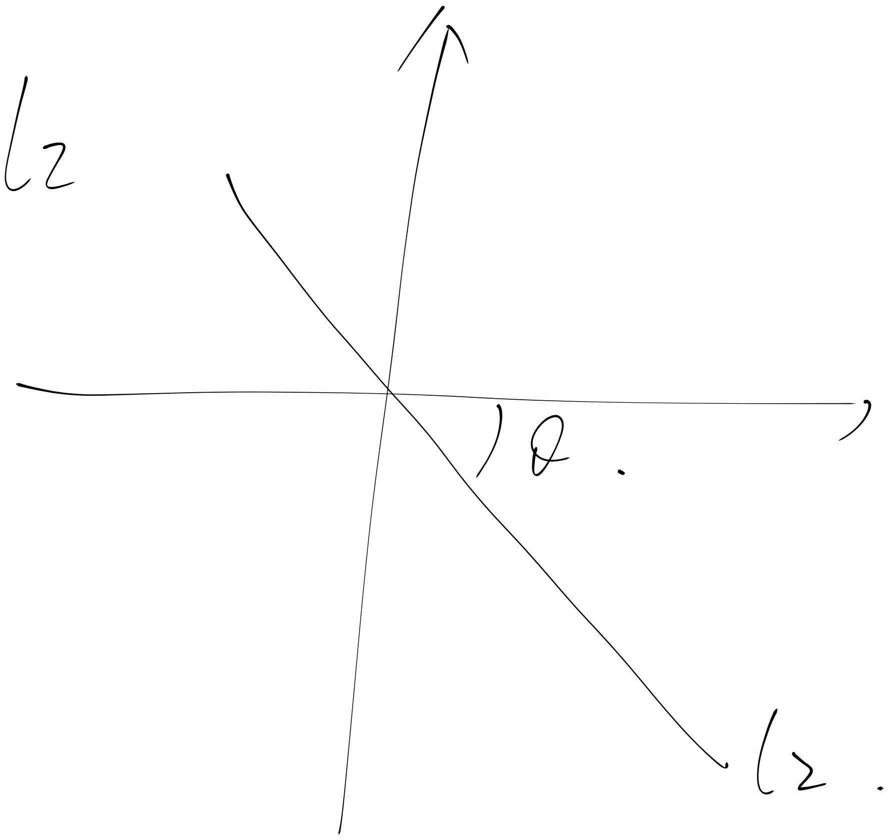
$$y_1 y_2 = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(-2\theta) & -\sin(-2\theta) \\ \sin(-2\theta) & \sin(-2\theta) \end{bmatrix}$$

is rotation clockwise by 2θ

(or counterclockwise by -2θ)

If l_2



then $y_2 = \begin{bmatrix} \cos(-2\theta), & \sin(-2\theta) \\ \sin(-2\theta), & -\cos(-2\theta) \end{bmatrix}$

so $y_1 y_2 = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$

is rotation counterclockwise by 2θ.

8. Let $D_4 = \{1, x, x^2, x^3,$
$y, xy, x^2y, x^3y\}$

$x^4 = 1, \quad y^2 = 1, \quad yxy^{-1} = x^{-1}$

Conjugacy classes of D_4

$\underset{1}{\{1\}}. \quad \underset{2}{\{x, x^3\}}, \quad \underset{1}{\{x^2\}}.$

$\underset{2}{\{y, x^2y\}}. \quad \underset{2}{\{xy, x^3y\}}.$

Divisors of 8 are, 1, 2, 4, 8.

So possible union of conjugacy classes containing 1 such that the order divides 8 are $\{1\}$, $\{1, x^2\}$.

$\{1, x^2, x, x^3\}$.

$\{1, x^2, y, x^2y\}$

$\{1, x^2, xy, x^3y\}$.

Direct check shows that these are all subgroups of D_4.