1. $D_4 = \left\{ \begin{matrix} 1, x, x^2, x^3, \\ y, xy, x^2y, x^3y \end{matrix} \right\}$.

The elements in the center form conjugacy classes consisting of one element.

So from the classification of conjugacy classes in D_4. the center of D_4 is

$$Z(D_4) = \{1, x^2\}.$$

2. $|G| = 99 = 9 \times 11$.

The number of Sylow 3-groups $s \mid 11$ and $s \equiv 1 \mod 3$, so $s = 1$.

The number of Sylow 11-group $s' \mid 9$ and $s' \equiv 1 \mod 11$, so $s' = 1$.

So both Sylow 3-group and 11-group are unique.

Let H be the Sylow 3-group. and K be the Sylow 11-group.

then $H \cap K = \{1\}$ and $G = HK$.

so $G \cong H \times K$,

$G \cong C_9 \times C_{11}$. or $C_3 \times C_3 \times C_{11}$

3. If $g \in Z(x)$, then $gxg^{-1} = (1\,2\,3)$

$(g(1), g(2), g(3)) = (1\,2\,3)$

so $g(1), g(2), g(3)$

$$\begin{array}{ccc} = 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{array}$$

$g(4), g(5) =$

$$\begin{array}{cc} 4 & 5 \\ 5 & 4 \end{array}$$

so $|Z(x)| = 3 \times 2$ and

$Z(x) = \{ 1, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 4 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix},$
$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix} \}$.

If $g\langle x\rangle g^{-1} = \langle x\rangle$.

then $gxg^{-1} = x$ or x^2

So $g(1), g(2), g(3)$

$$
\begin{array}{cccc}
= & 1 & 2 & 3 \\
 & 2 & 3 & 1 \\
 & 3 & 1 & 2 \\
 & 1 & 3 & 2 \\
 & 3 & 2 & 1 \\
 & 2 & 1 & 3
\end{array}
$$

$$
\begin{array}{cc}
g(4), g(5) = & 4, 5 \\
 & 5\ 4
\end{array}
$$

$|N(x)| = 12$. $N(x) = \Big\{ 1, \begin{pmatrix}1&2&3&4&5\\2&3&1&4&5\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\3&1&2&4&5\end{pmatrix}$

$\begin{pmatrix}1&2&3&4&5\\1&3&2&4&5\end{pmatrix} \begin{pmatrix}1&2&3&4&5\\3&2&1&4&5\end{pmatrix}$

$\begin{pmatrix}1&2&3&4&5\\2&1&3&4&5\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\1&2&3&5&4\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\3&1&2&5&4\end{pmatrix}$

$\begin{pmatrix}1&2&3&4&5\\1&3&2&5&4\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\3&2&1&5&4\end{pmatrix} \begin{pmatrix}1&2&3&4&5\\2&1&3&5&4\end{pmatrix}, \begin{pmatrix}1&2&3&4&5\\2&1&3&5&4\end{pmatrix} \Big\}$

4. In the solution of practice exam.

5. Let $G/Z = \langle xZ \rangle$.

then $G = \bigcup_{k=0}^{+\infty} x^k Z$

$\forall g \in G$, $g = x^k \cdot h$, $h \in Z$.

$\forall h_1, h_2 \in Z(G)$, $(x^{k_1} h_1 \cdot x^{k_2} h_2) = x^{k_1} x^{k_2} h_1 h_2$

$= (x^{k_2} h_2)(x^{k_1} h_1)$

so $Z(G) = G$.

6. If $p = q$, $G = C_p \times C_p$ or C_{p^2}.

both contains a subgroup of order p as normal subgroup.

If $p \neq q$, assume $p < q$, then

Sylow q-subgroup is unique, hence a normal subgroup.

7. $20 = 2^2 \times 5$

so the number of Sylow 5-group is 1.

denote H to be the Sylow 5-group.

then every element of order 5 is contained in H. $H = \langle x \rangle$.

all the order-five elements are

x, x^2, x^3, x^4

So there're 4 such elements.