1. Associativity.

$$\big((h_1,k_1)(h_2,k_2)\big)(h_3,k_3)$$

$$= (h_1\varphi(k_1)(h_2),\ k_1k_2)(h_3,k_3)$$

$$= \big(h_1\ \varphi(k_1)(h_2)\cdot\varphi(k_1k_2)(h_3),\ k_1k_2k_3\big)$$

$$= \Big(h_1\ \varphi(k_1)(h_2)\cdot\varphi(k_1)\big(\varphi(k_2)(h_3)\big),\ k_1k_2k_3\Big)$$

$$= \big(h_1\ \varphi(k_1)(h_2\cdot\varphi(k_2)(h_3)),\ k_1h_2k_3\big)$$

$$(h_1,k_1)\cdot\big((h_2,k_2)(h_3,k_3)\big)$$

$$= (h_1,k_1)\cdot(h_2\varphi(k_2)(h_3),\ k_2k_3)$$

$$= \big(h_1\cdot\varphi(k_1)(h_2\varphi(k_2)(h_3)),\ k_1k_2k_3\big)$$

So $\big((h_1,k_1)(h_2,k_2)\big)(h_3,k_3) = (h_1,k_1)\big((h_2,k_2)(h_3,k_3)\big)$

Identity.

$(1,1)\cdot(h,k)$

$= (1\cdot\varphi(1)(h), 1\cdot k) = (h, k)$

$(h,k)(1,1)$

$= (h\cdot\varphi(k)(1), k\cdot 1) = (h,k)$.

Inverse: $\left(\text{Idea: } (hk)^{-1} = k^{-1}h^{-1} = \underbrace{(k^{-1}h^{-1}k)}_{\in H}\underbrace{k^{-1}}_{\in K}\right)$

Let $g = (\varphi(k^{-1})(h^{-1}), k^{-1})$

$g\cdot(h,k) = (\varphi(k^{-1})(h^{-1})\cdot\varphi(k^{-1})(h), k^{-1}k)$

$= (\varphi(k^{-1})(h^{-1}\cdot h), 1)$

$= (1, 1)$

$(h,k)\cdot g = (h\cdot\varphi(k)(\varphi(k^{-1})(h^{-1})), k\cdot k^{-1})$

$$= \left(h \cdot \left(\varphi(k) \circ \varphi(k^{-1})\right)(h^{-1}),\ k k^{-1}\right)$$

$$= \left(h\ \varphi(k k^{-1})(h^{-1}),\ k k^{-1}\right)$$

$$= (h \cdot h^{-1},\ 1) = (1,\ 1)$$

so g is the inverse of (h, k)

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2. $55 = 5 \times 11$.

The number of Sylow 11-group divides 5 and $\equiv 1 \mod 5$.

so Sylow 11-group is unique, denoted by H. $H \trianglelefteq G$.

Let K be a sylow 5-group.

Then $H \cap K = \{1\}$. $G = HK$.

Let $H = \langle x \rangle$. $K = \langle y \rangle$.

then $x^{11}=1$. $y^5=1$. $yxy^{-1}=x^r$.

$y^5xy^{-5}=x$ and $y^5xy^{-5}=x^{r^5}$

so $r^5\equiv 1 \mod 11$.

$r = \bar{0}\ \bar{1}\ \bar{2}\ \bar{3}\ \bar{4}\ \bar{5}\ \bar{6}\ \bar{7}\ \bar{8}\ \bar{9}\ \bar{10}$

$r^5 = \bar{0}\ \bar{1}\ \bar{10}.\ \bar{1},\ \bar{1},\ \bar{1},\ \bar{10},\ \bar{10}.\ \bar{10},\ \bar{1},\ \bar{10}$.

so $r\equiv 1.\ 3,\ 4,\ 5,\ 9 \mod 5$.

If $r\equiv 1 \mod 5$. then $yxy^{-1}=x$.

$$G\cong H\times K$$

If $r\equiv 3,4,5,9 \mod 5$,

then $\varphi: K \to \mathrm{Aut}(H)$ is not trivial.
$k \mapsto \varphi(k): h \mapsto khk^{-1}$

and all different r can be choosen to be 3

by choosing different generator for K.

So this gives the other isomorphism class of $G = \langle x, y \rangle$ $x^{11}=1$, $y^5=1$

$yxy^{-1} = x^3$

($\mathrm{Aut}(H) \cong (\mathbb{Z}/11\mathbb{Z})^\times \cong (\mathbb{Z}/10\mathbb{Z})$.

There is only one nontrivial homomorphism from $C_5 \to C_{10}$ up to the choice of generator for C_5)