HW 6 Solutions

1. The units in $\mathbb{Z}/n\mathbb{Z}$ are represented by integers coprime to n. Therefore,

$$(\mathbb{Z}/12\mathbb{Z})^* = \{\overline{1}, \overline{5}, \overline{7}, \overline{11}\}, \text{ and } (\mathbb{Z}/8\mathbb{Z})^* = \{\overline{1}, \overline{3}, \overline{5}, \overline{7}\}.$$

2. (From student.)
   Let U be a set. Let $R = \mathcal{P}(U)$ be the set of subsets of U. Consider the map

$$\begin{array}{cccl} \varphi: & R & \to & \prod_{u\in U} \mathbb{Z}/2\mathbb{Z} \\ & A & \mapsto & (x_u)_{u\in U} \text{ where } \begin{cases} x_u = 1 \text{ if } u \in A \\ x_u = 0 \text{ if } u \notin A. \end{cases} \end{array}$$

   This is a bijection since its inverse is given by

$$\begin{array}{cccc} \psi: & \prod_{u\in U} \mathbb{Z}/2\mathbb{Z} & \to & R \\ & (x_u)_{u\in U} & \mapsto & \{u \in U : x_u = 1\}. \end{array}$$

   Under ψ, the multiplication on the ring $\prod_{u\in U} \mathbb{Z}/2\mathbb{Z}$ corresponds to the operation $A \cdot B := A \cap B$ on R. This is because for any $x = (x_u)_{u\in U}$ and $y = (y_u)_{u\in U}$ in $\prod_{u\in U} \mathbb{Z}/2\mathbb{Z}$, we have

$$\psi(x \cdot y) = \psi(x) \cap \psi(y),$$

   since

$$\begin{aligned} u \in \psi(x \cdot y) \quad & \Leftrightarrow u \in U \text{ satisfies } (x \cdot y)_u = 1 \\ & \Leftrightarrow u \in U \text{ satisfies } x_u \cdot y_u = 1 \\ & \Leftrightarrow u \in U \text{ satisfies } x_u = 1 \text{ and } y_u = 1 \\ & \Leftrightarrow u \in U \text{ satisfies } u \in \psi(x) \text{ and } u \in \psi(y) \\ & \Leftrightarrow u \in \psi(x) \cap \psi(y). \end{aligned}$$

   Similarly, the addition on the ring $\prod_{u\in U} \mathbb{Z}/2\mathbb{Z}$ corresponds to the operation $A + B := A \cup B - A \cap B$ on R. Therefore, R is a ring under the above two operations.

3. (a.) We have $q(x) = x + 3$ and $r(x) = 5$, since

$$2x^3 + 7x^2 + 4x + 8 = (2x^2 + x + 1)(x + 3) + 5 \text{ in } \mathbb{Z}[x].$$

(b.) We have $q(x) = 4x + 5$ and $r(x) = 1$, since

$$2x^2 + 2x = (2x + 1)(4x + 5) + 1 \text{ in } (\mathbb{Z}/6\mathbb{Z})[x].$$

Note that $q(x) = 4x + 2$ and $r(x) = 4$ also works.

(c.) We have $q(x) = 4x + 2$ and $r(x) = 4$, since

$$2x^2 + 2x = (5x + 1)(4x + 2) + 4 \text{ in } (\mathbb{Z}/6\mathbb{Z})[x].$$