HW 8 Solutions

March 25, 2019

1. Let R be a finite integral domain. Let $0 \neq a \in R$. Then

$$\varphi : R \to R : r \mapsto ar$$

injects: if $r, s \in R$ satisfy $ar = as$, then $a(r - s) = 0 \Rightarrow r - s = 0$ since $a \neq 0$ and R is an integral domain.

Next, since R is finite and φ injects, φ must also surject. Therefore $1 = ar$ for some $r \in R$.

2. (a) The inclusion $IJ \subset I \cap J$ is clear. For the reverse inclusion, note that $I + J = R$ implies that

$$I \cap J = R \cdot (I \cap J) \subset (I + J) \cdot (I \cap J) \subset (I + J)I \cap (I + J)J \subset IJ.$$

(b) Let $a, b \in R$ be given. We need to show that there exists $x \in R$ such that $x \equiv a \pmod{I}$ and $x \equiv b \pmod{J}$.

Let $e_1 \in I, e_2 \in J$ satisfy $1 = e_1 + e_2$. Set $x = be_1 + ae_2$. Then $x \equiv a \pmod{I}$ and $x \equiv b \pmod{J}$ since

$$x+I = be_1+ae_2+I = be_1+a(1-e_1)+I = a+(b-a)e_1+I = a+I \text{ in } R/I$$

since $e_1 \in I$. Similarly, we get $x \equiv b \pmod{J}$.

(c) The ring homomorphism

$$R \to R/I \times R/J : r \mapsto (r \pmod{I}, r \pmod{J})$$

has kernel $I \cap J = IJ = 0$ (the first equality is by part (a)). This map surjects by part (b). By the first isomorphism theorem, we have

$$R \cong R/I \times R/J.$$

(d) Let $e_1 \in I, e_2 \in J$ satisfy $1 = e_1 + e_2$ as above. Then under the assumption that $IJ = 0$, the elements e_1, e_2 are the idempotents corresponding to the product decomposition $R \cong R/I \times R/J$ since we have

- $e_1 e_2 \in IJ = 0$.
- $e_1^2 = e_1(1 - e_2) = e_1 - e_1 e_2 = e_1$.
- Similarly $e_2^2 = e_2$.

3. Note that $f(x) = \frac{x^5-1}{x-1}$ implies that $\alpha^5 - 1 = 0$ in $\mathbb{Z}[\alpha] = \mathbb{Z}[x]/(f)$. Therefore
$$(\alpha^3 + \alpha^2 + \alpha)(\alpha^5 + 1) = 2(\alpha^3 + \alpha^2 + \alpha).$$

4. Let J be a maximal ideal of $\mathbb{C}[x, y]/(x, y)$. By the correspondence theorem, this corresponds to an *maximal* ideal I of $\mathbb{C}[x, y]$ containing (xy). Since I is maximal, by Hilbert's nullstellensatz, $I = (x-a, y-b)$ for some $a, b \in \mathbb{C}$.

   This means that I consists of polynomials $f(x, y)$ passing through the point (a, b). The condition $xy \in I$ then tells us that $g(x, y) = xy$ is one such polynomial passing through the point (a, b), so $g(a, b) = ab = 0$. Therefore $I = (x - a, y - b)$ contains the element xy iff $a = 0$ or $b = 0$. By applying the correspondence theorem again, we conclude that the maximal ideals of $\mathbb{C}[x, y]/(x, y)$ are of the form $(x - a, y - b) + (x, y)$, where $a = 0$ or $b = 0$.

   Note. A more precise way to say the previous paragraph is the following. The ideal I is the kernel of the evaluation map $\varphi : \mathbb{C}[x, y] \to \mathbb{C} : f(x, y) \mapsto f(a, b)$. Therefore, $xy \in I = \ker \varphi$ iff $\varphi(xy) = ab$ is zero.

5. In this and the next solution, we use the following fact.

   Fact. Let $n \geq 0$ be an integer. Then $\mathbb{Z}/(n)$ is a field iff n is a prime. □

   Since
$$\frac{\mathbb{Z}[i]}{(i-2)} \cong \frac{\mathbb{Z}[x]}{(x-2, x^2+1)} \cong \frac{\mathbb{Z}[2]}{(0, 2^2+1)} \cong \frac{\mathbb{Z}}{(5)}$$
   is a field, $(i - 2)$ is a maximal ideal of $\mathbb{Z}[i]$.

6. Since
$$\frac{\mathbb{Z}[i]}{(i+3)} \cong \frac{\mathbb{Z}[x]}{(x+3, x^2+1)} \cong \frac{\mathbb{Z}[-3]}{(0, (-3)^2+1)} \cong \frac{\mathbb{Z}}{(10)}$$
   is not a field, $(i + 3)$ is not a maximal ideal of $\mathbb{Z}[i]$.