HW 9 Solutions

March 29, 2019

1. Since x^3+x+1 is a degree 3 polynomial with no roots in $\mathbb{F}_2$, it is irreducible in $\mathbb{F}_2[x]$. Since $\mathbb{F}_2[x]$ is a PID, (f) is a maximal ideal of $\mathbb{F}_2[x]$ and so $\mathbb{F}_2[x]/(f(x))$ is a field.

   On the other hand, $x^3 + x + 1$ has the root $x = 1$ in $\mathbb{F}_3$, so $\mathbb{F}_3[x]/(f(x))$ is not a field.

2. We have

   $$\frac{-4}{2+i} = \frac{-4}{2+i} \cdot \frac{2-i}{2-i} = \frac{-8+4i}{5} = -1.6 + 0.8i \approx 2 + i$$

   so

   $$-4 = (2+i)(2+i) + r,\ r = -1$$

   satsifies $\sigma(-1) = 1 < \sigma(2+i) = 5$.

3. Since a, b are associates in R, there exists a unit $c \in R$ such that $a = bc$. Let $b = xy$ for some $x, y \in R$. Then $a = xyc = x(yc)$. Since a is irreducible, either x is a unit or yc is a unit. If x is a unit, we are done. Otherwise yc is a unit. Since c is a unit, so is y, so we are again done. Therefore b is irreducible.

4. We claim that (x, y) is not a principal ideal of $\mathbb{C}[x, y]$. For the sake of contradiction, suppose that $(x, y) = (f(x, y))$ for some $f \in \mathbb{C}[x, y]$. Then $x = g(x, y)f(x, y)$ for some $g(x, y) \in \mathbb{C}[x, y]$.

   - Since
     $$0 = \deg_y x = \deg_y g(x, y) + \deg_y f(x, y),$$
     we see that $\deg_y f(x, y) = 0$ so $f(x, y) = f(x) \in \mathbb{C}[x]$.
   - Next, $1 = \deg_x x = \deg_x g(x, y) + \deg_x f(x)$ so $\deg_x f(x) = 0, 1$.
     - If $\deg_x f(x) = 0$, then $f \in \mathbb{C}$.
     - Otherwise $\deg_x f(x) = 1$, so $f = ax + b$ for some $a, b \in F$ and $a \neq 0$. But x is a multiple of f only if $b = 0$. So $f(x) = ax$.

   Therefore, from the fact that $x \in (f)$, we get that either $f \in \mathbb{C}$ or $f = ax$ for some $a \in \mathbb{C}$.

Similarly, from $y \in (f)$, we get either $f \in \mathbb{C}$ or $f = cy$ for some $c \in \mathbb{C}$.

Therefore $f \in \mathbb{C}$. But now $(f) = \mathbb{C}[x,y]$ whereas $(x,y) \neq \mathbb{C}[x,y]$, a contradiction.