Math 371 **Name:** ______________________
Spring 2019
Midterm 1
02/19/2019
Time Limit: 80 Minutes ID ______________________

"My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this"

Signature ______________________

This exam contains 12 pages (including this cover page) and 11 questions.
Total of points is 110.

- Check your exam to make sure all 12 pages are present.
- You may use writing implements and a single handwritten sheet of 8.5"x11" paper.
- NO CALCULATORS.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
Total:	110	

1. (10 points) Let H be a subgroup of G. State the definition of normalizer of H in G. Find the normalizer of $H = \{1, (123), (132)\}$ in S_4.

$N(H) = \{g \in G \mid gHg^{-1} = H\}$

$g(123)g^{-1} = (g(1).g(2).g(3))$.

Since (123) and (132) are both generators of H,

$g \in N(H)$ if and only if

$g(123)g^{-1} = (123)$ or (132)

So $\{g(1), g(2), g(3)\} = \{1, 2, 3\}$.

$N(H) = \left\{ 1, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix} \right\}$

2. (10 points) Write the element $(123)(2345) \in S_5$ as product of disjoint cycles.

1 2 3 4 5.

1 3 4 5 2 $= (12)(345)$

2 1 4 5 3

3. (10 points) Find the Sylow 2-subgroups of D_6.

$$D_6 = \{1, x, x^2, x^3, x^4, x^5, y, xy, x^2y, x^3y, x^4y, x^5y\}$$

Any subgroup of D_6 is either cyclic group or dihedral group.

Since there is no element with order 4. so the Sylow 2-subgroup of D_6 must be D_2. generated by a rotation of angle π and a reflection.

So the Sylow 2-subgroups are:

$\{1, x^3, y, x^3y\}$

$\{1, x^3, xy, x^4y\}$

$\{1, x^3, x^2y, x^5y\}$

4. (10 points) Find all the normal subgroups of S_4.

$|S_4| = 4 \times 3 \times 2 \times 1 = 24$.

All the conjugacy classes are

C_1 $\{1\}$. 1

C_2 $\{(12), (23), (24), (13), (14), (34)\}$. 6

C_3 $\{(123), (132), (124), (134), (234), (243), (143), (142)\}$ 8

C_4 $\{(1234), (1432), (1324), (1243), (1423), (1342)\}$ 6.

C_5 $\{(12)(34), (13)(24), (14)(23)\}$. 3

Normal subgroups are union of conjugacy classes containing $\{1\}$. So all the possible unions are

1, $1+3$, $1+3+8$, $1+3+6+8+8$.

$\{1\}$, $C_1 \cup C_5$, $C_1 \cup C_5 \cup C_3$. S_4.

5. (10 points) Let

$$H = \{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\}$$

be a subgroup of special orthogonal group $SO(2)$. Prove that the quotient group $SO(2)/H$ is isomorphic to $SO(2)$.

Define a map.

$\rho: SO(2) \to SO(2)$.

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \mapsto \begin{bmatrix}\cos 2\theta, & -\sin 2\theta\\ \sin 2\theta, & \cos 2\theta\end{bmatrix}.$$

Then ρ is a homomorphism.

and $\ker\rho = \left\{\begin{bmatrix}1 & \\ & 1\end{bmatrix}, \begin{bmatrix}-1 & \\ & -1\end{bmatrix}\right\} = H$

(or define $\rho: x \mapsto x^2$).

ρ is surjective.

so $SO(2)/H \cong SO(2)$

6. (10 points) Let $C(G)$ be the center of G. Prove that $|G/C(G)|$ can not be 15.

Assume $|G/C(G)| = 15$.

Denote $\bar{G} = G/C(G)$.

Then $\bar{G}$ has a unique Sylow 5-subgroup H and a unique Sylow 3-subgroup K.

So $H \cap K = \{1\}$. $H \lhd \bar{G}$. $K \lhd \bar{G}$.

$HK = \bar{G}$ and $\bar{G} \cong H \times K$

$\cong C_5 \times C_3$.

$\cong C_{15}$

So $G = \bigcup_{k=0}^{14} x^k C(G)$ for any $g_1, g_2 \in C(G)$

$(x^{k_1} g_1)(x^{k_2} g_2) = x^{k_1+k_2} g_1 g_2 = (x^{k_2} g_2)(x^{k_1} g_1)$

So G is abelian. and $C(G) = G$

7. (10 points) Prove that a group of order 56 is not a simple group.

$|G| = 56 = 2^3 \times 7$

Let s be the number of Sylow 7-group.

$s | 8$. $s \equiv 1 \mod 7$.

so $s = 1$. or, 8

If $s = 1$. then Sylow 7-group is normal.

If $s = 8$, let $K_1 \dots K_8$ be all the Sylow 7-groups. $K_i \cap K_j = \{1\}$ for $i \neq j$.

So $\bigcup_{i=1}^{8} K_i = 6 \times 8 + 1 = 49$.

$|G - \bigcup_{i=1}^{8} K_i| = 7$.

Let H be a Sylow 2-group. $H \cap K_i = \{1\}$

So $H \subset \{1\} \cup (G - \bigcup_{i=1}^{8} K_i)$ and $|H| = 8$.

$H = \{1\} \cup (G - \bigcup_{i=1}^{8} K_i)$. Such H is unique

So H is a normal subgroup.

8. (10 points) Classify all finite groups of order 14.

$|G| = 14$.

Let s be the number of Sylow 7-subgroups

then $s \equiv 1 \mod 7$. $s \mid 2$.

so $s = 1$.

Let H be the unique Sylow 7-subgroup.

$H \triangleleft G$.

Let K be a Sylow 2-subgroup.

$H = \langle x \rangle$, $K = \langle y \rangle$.

$x^7 = 1$. $y^2 = 1$.

$yxy^{-1} = x^j$. $j^2 \equiv 1 \mod 2$

$\Rightarrow$ j	1	2	3	4	5	6
j^2	1	4	2	2	4	1

so $j = 1$ or $j = 6$.

Case 1, $j=1$. $G \cong C_2 \times C_7$.

Case 2, $j=5$. $G = \langle x, y \rangle$. $x^7 = 1$. $y^2 = 1$. $yxy^{-1} = x^6$ D_7.

9. (10 points) Is there a transitive operation of S_4 on a set of five elements? Why?

No, If there is such an operation.

then $|G| = |S| \cdot |G_x|$.

$x \in S$. $|S| = 5$.

$|G| = 4 \times 3 \times 2 \times 1 = 24$

$5 \nmid 24$. contradiction.

10. (10 points) Classify finite groups of order 28.

Let s be the number of Sylow 7-subgroups.

$s \mid 4$ and $s \equiv 1 \mod 7$.

so $s=1$. Let H be the unique Sylow 7-group.

Let K be a Sylow 2-group.

Then $G \cong H \rtimes K$.

Case 1. If $K \cong C_2 \times C_2$.

Let $H = \langle x \rangle$, $K = \langle y_1, y_2 \rangle$.

$x^2 = y_1^2 = y_2^2 = 1$.

$y_i x y_i^{-1} = x^{j_i}$, Then $y_i^2 x y_i^{-2} = x^{j_i^2} = x$

$j_i^2 \equiv 1 \ (\mathrm{mod}\ 2)$.

$j_i = 1$ or 6, $(j_1, j_2) = (1,1)$. $G \cong C_2 \times C_2 \times C_7$.

$(j_1, j_2) = (1,6)$ or $(6,1)$. $(6,6)$, we can choose different generators for $C_2 \times C_2$ to make $(j_1, j_2) = (1,6)$. so $G \cong C_2 \times D_7 \cong D_{14}$.

Case 2: If $K \cong C_4$

$H = \langle x \rangle . \quad K = \langle y \rangle$

$x^2 = y^4 = 1 .$

$yxy^{-1} = x^j , \quad j^4 \equiv 1 \pmod 7$

$j = 1 , \text{ or } 6 .$

$j = 1 , \quad G \cong C_4 \times C_7$

$j = 6 \quad G = \langle x , y \rangle . \ x^2 = y^4 = 1 ,$

$yxy^{-1} = x^6 .$

Four different isomorphism classes.

Bonus Question

11. (10 points) Let p be a prime number and G be a p-group. Let H be a proper subgroup of G (a subgroup of G which is not equal to G). Prove that the normalizer $N(H)$ is strictly larger than H. (Hint: Restrict the operation of G on the cosets G/H to H).

Let $|G| = p^n$, $|H| = p^k$, $0 \le k < n$.

if $k = 0$, $H = \{1\}$. $N(H) = G$.

If $1 \le k < n$, $|G/H| = p^{n-k}$.

Restrict the action of H on G/H.

$$G/H = \bigsqcup_{i=1}^{m} O_i$$

Assume $O_1 = \{[H]\}$. $|O_1| = 1$.

since $|G/H| = |O_1| + \dots |O_m| \equiv 0 \pmod p$

$$|H| = |O_i| \cdot |H_{x_i}| = p^k$$

and $x_i \in O_i$.

Then $|O_i| = 1$ or p^{k_i}, $k_i > 0$.

So $\exists$ O_i other than O_1, s.t.

$$|O_i| = 1$$

$Hx_i = H$, Let $x_i = g_i H$, $g_i \notin H$.

then $\forall$ $h \in H$.

$$h g_i H = g_i H.$$

so $g_i^{-1} h g_i \in H$.

so $g_i^{-1} \in N(H)$.

So $N(H) \neq H$.