MATH 370 **FINAL** 12/14/18

You may *not* use your book, computer, calculator, cell phone, smartwatch or any other device on this exam. You are required to show your work on each problem on this exam (except problem 1). The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Score	Problem	Score
1		6	
2		7	
3		8	
4		9	
5		10	
Total:			

My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this examination:

Name (printed) ________________________________

Signature ________________________________ **Date** ______________

1. (20 points) Circle **True** or **False** (No explanation needed):

   (a) Any two abelian groups of order 6 are isomorphic. True / False

   (b) In a group, each linear equation $a * x = b$ has a unique solution. True / False

   (c) There may be a group in which associative law fails. True / False

   (d) Any cyclic group has two subgoups of order 2. True / False

   (e) The order a subgroup always divide the order of the group. True / False

   (f) $\mathbb{R}$ under multiplication is a group. True / False

   (g) Every cyclic group of order greater than 2 has at least two generators. True / False

   (h) Every group is isomorphic to some group of permutations. True / False

   (i) Any abelian group is generated by finitely many elements. True / False

   (j) There is only one cyclic group of a given order, up to isomorphism. True / False

2. Let H and K be subgroups of G.

   (a) (10 points) Prove that $H \cap K$ is a subgroup of G.

   (b) (10 points) Find an example that $H \cup K$ is not a subgroup.

3. Consider the symmetric group S_n and the subgroup A_n of even permutations.

   (a) (10 points) Prove that

   $$\alpha \sim \beta \;\Leftrightarrow\; \alpha\beta \in A_n$$

   is an equivalence relation on S_n.

   (b) (5 points) Find the partition on S_n corresponding to the equivalence relation above.

   (c) (10 points) Find a non-trivial homomorphism from S_n to $\mathbb{Z}_2$.

4. (15 points) Find an integer x such that $x = 1 \mod 3, x = 2 \mod 7$ and $x = 4 \mod 10$.

5. (15 points) Find all abelian groups (up to isomorphism) of order 360.

6. (15 points) Determine the number of elements of order 15 in $\mathbb{Z}_{30} \times \mathbb{Z}_{25}$

7. (a) (15 points) Find all generators of $\mathbb{Z}_{54}$.

(b) (15 points) Let G be a cyclic group of order n and let m be an integer which is relatively prime to n. Show that the map $\phi(a) = a^m$ is an automorphism on G.

8. Let

$$N = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{Z}_5 \right\}$$

be a subset of the vectorspace $M_{2\times 2}(\mathbb{Z}_5)$ over $\mathbb{Z}_5$.

(a) (10 points) Show that N is a subspace of $M_{2\times 2}(\mathbb{Z}_5)$.

(b) (15 points) Show that the following set is linearly dependent over $\mathbb{Z}_5$

$$\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix} \right\}$$

(c) (20 points) Find a, b and c in $\mathbb{Z}_5$ such that

$$\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} a & b \\ c & a \end{pmatrix} \right\}$$

form a basis of N.

9. Find an example of the following property with a brief explanation:

   (a) (5 points) an infinite group in which every element has finite order, and for each positive integer n there is an element of order n.

   (b) (5 points) a group with an element of infinite order and an element of order 2.

   (c) (5 points) a group G such that every finite group is isomorphic to some subgroup of G.

   (d) (5 points) a non cyclic group such that every subgroup is cyclic.

10. Let G be a group and $Z(G) = \{x \in G \mid xg = gx \text{ for all } g \in G\}$ be the center of G.

   (a) (15 points) Prove that if $G/Z(G)$ is cyclic then G is abelian.

   (b) (20 points) Find an example such that $G/Z(G)$ is abelian but G is not an abelian group.

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