Math 371 Homework#2

Due on 1/31 at the beginning of Lecture

1. Let G be the subset of $\mathrm{GL}(2,\mathbb{R})$ consisting of matrices

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

with $a^2 + b^2 \neq 0$. Prove that G is a subgroup of $\mathrm{GL}(2,\mathbb{R})$ and isomorphic to $\mathbb{C}^\times$.

2. Prove that the map $a \mapsto a^{-1}$ is a group isomorphism of G to itself if and only if G is abelian.

3. **Second Isomorphism Theorem**. Let H be a normal subgroup of group G and K be a subgroup of G. Prove
   (a) $HK = \{hk | h \in H, k \in K\}$ is a subgroup of G.
   (b) $H \cap K$ is a normal subgroup of K.
   (c) There is an isomorphism $HK/H \cong K/H \cap K$.

4. Determine the order of the group of rotational symmetries of a cube.

5. Let $O_1, \cdots, O_k$ be all the conjugacy classes in a finite group G. Choose $x_i \in O_i$ and let $C_i = \{g \in G | g x_i g^{-1} = x_i\}$ (which is called the centralizer of x_i). Denote $n_i = |C_i|$. Prove

$$\frac{1}{n_1} + \frac{1}{n_2} + \cdots \frac{1}{n_k} = 1$$

6. **Artin, Chapter 6, 8.1** Does the rule $P * A = PAP^t$ define an operation of $\mathrm{GL}(n)$ on the set of $n \times n$ matrices? (Here P^t means the transpose of P)

7. **Artin, Chapter 6, 11.1** Describe all ways in which S_3 can operate on a set of four elements.