Math 371 Homework#3

Due on 2/7 at the beginning of Lecture

1. Prove that a subgroup H of G is normal if and only if $Hg = gH$ for any $g \in G$. Here $gH = \{gh|h \in H\}$ and $Hg = \{hg|h \in H\}$.

2. Let H be a subgroup of G with $|G/H| = 2$. Prove that H is a normal subgroup.

3. Prove that a normal subgroup H of G is the union of some conjugacy classes in G.

4. The 2-cycles (i_1, i_2) in symmetric group S_n are called transpositions. Prove that every element $x \in S_n$ can be written as a product of transpositions. (Hint: use induction on $|S|$ where $S = \{i \in \{1, \cdots, n\}|x(i) \neq i\}$.)

5. In this question, you will classify all the normal subgroups of S_4.
   (a) How many conjugacy classes are there in S_4?
   (b) List all the elements in each conjugacy class.
   (c) Find possible subsets G of S_4 such that
      i. G contains identity,
      ii. G is the union of some conjugacy classes,
      iii. $|G|$ divides $|S_4|$.
   (d) Find all normal subgroups of S_4 (based on problem 3 and problem 4).

6. Prove
   (a) Any subgroup of a cyclic group C_n is still a cyclic group.
   (b) Any subgroup of dihedral group D_n is either a cyclic group or a dihedral group.

7. Let $y_1, y_2 \in \mathrm{O}(2)$ be two reflections about lines l_1, l_2. Assume the angle between l_1 and l_2 is θ. Find all the possible compositions $y_1 y_2$.

8. Find all the normal subgroups of D_4. (Hint: use the procedure described in problem 5.)