Math 371 Homework#8

Due on 3/21 at the beginning of Lecture

1. **Artin Chapter 11, 7.1** Prove that a finite domain is a field. (Hint: use the hint of Q1 in homework 6)

2. **Artin, Chapter 11, 6.8** Let I and J be ideals of a ring R such that $I + J = R$.

   (a) Prove that $IJ = I \cap J$.

   (b) Prove the *Chinese Remainder Theorem*: For any pair a, b of elements of R, there is an element x such that $x \equiv a \mod I$ and $x \equiv b \mod J$. (Here the notation $x \equiv a \mod I$ means $x - a \in I$.

   (Hint: Since $I + J = R$, we have $1 = e_1 + e_2$ with $e_1 \in I$ and $e_2 \in J$. Consider the case $(a, b) = (1, 0)$ or $(0, 1)$ by using e_1, e_2. Try the product of e_i with a or b to solve the case $(a, b) = (a, 0)$ and $(a, b) = (0, b)$. Combining the solved cases to solve the general case (a, b))

   (c) Prove that if $IJ = 0$, then R is isomorphic to the product ring $R/I \times R/J$. (Hint: consider the natural homomorphism from $R \to R/I \times R/J$.

   (d) Describe the idempotents corresponding to the product decomposition in (c).

3. **Artin Chapter 11, 5.1** Let $f = x^4 + x^3 + x^2 + x + 1$ and let α denote the residue of x in the ring $R = \mathbb{Z}[x]/(f)$. Express $(\alpha^3 + \alpha^2 + \alpha)(\alpha^5 + 1)$ in terms of the basis $(1, \alpha, \alpha^2, \alpha^3)$ of R.

4. Use $Hibert'sNullstellensatz$ to determine all the maximal ideals in $\mathbb{C}[x, y]/(xy)$.

5. Is $(i - 2)$ a maximal ideal in the ring of Gaussian integers $\mathbb{Z}[i]$? Why?

6. Is $(i + 3)$ a maximal ideal in the ring of Gaussian integers $\mathbb{Z}[i]$? Why?