Lecture 1. Groups (chapter 2)

- Definition and Examples
- Subgroups
- Homomorphism
- Quotient Groups.

1. Groups

Def: A "Law of composition" (or "binary operation") $*$ on a set S is a rule for asigning each ordered pair (a, b) $(a \in S, b \in S)$ an element c of S.

$$* : S \times S \to S$$
$$(a, b) \mapsto a * b.$$

Ex: $(\mathbb{Z}^+, +)$. $\quad \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$

$$(a, b) \mapsto a+b.$$

Nonex: $(\mathbb{Z}^+, -)$

Def: (Associativity) A binary operation $(S, *)$ is "associative" if $(a*b)*c = a*(b*c)$

Ex: $(\mathbb{Z}, +)$

Nonex: $(\mathbb{Z}, -)$.

Def: A group $(G, *)$ is a set with binary operation satisfying the following properties. (write $ab = a*b$).

- Associativity $(ab)c = a(bc)$
- Identity element $1 \in G$, $1 \cdot a = a$ and $a \cdot 1 = a$.
- Inverse: $\forall a \in G$, $\exists b \in G$ such that $a \cdot b = b \cdot a = 1$.

Ex: Permutation group. symmetric group of n-elements.

$$S_n = \{ x : \{1, \cdots n\} \to \{1 \cdots n\} \}.$$

Ex: General linear group.

$GL(n, \mathbb{R}) = \{ n \times n \text{ invertible matrices } A \}$.

$GL(n, \mathbb{C})$

· Subgroup.

Def: A subset S of a group G is a subgroup if.

- closure: $a \cdot b \in S$. $a, b \in S$

- Identity $1 \in S$
- Inverse: if $a \in S$, then $a^{-1} \in S$

Ex: $\{x \in S_n \mid x(n) = n\} \overset{1:1}{\longleftrightarrow} S_{n-1} \subset S_n$

Ex: $\{x \in GL(n) \mid \det x = 1\} \subset GL(n)$

(special linear group) $= SL(n)$

Non ex: $\mathbb{Z}^+ \subset \mathbb{Z}$

Normal subgroup

Def: A subgroup H of G is normal if $\forall g \in G, h \in H.\ ghg^{-1} \in H$

Ex: $SL(n) \subset GL(n)$ Non Ex: $S_{n-1} \subset S_n$.

Homomorphism:

Def: A homomorphism $\varphi: G \to G'$ is a map from G to G'. s.t. $\forall a, b \in G.\ \varphi(ab) = \varphi(a) \cdot \varphi(b)$

Ex: $GL(n, \mathbb{R}) \longrightarrow \mathbb{R}^\times = \mathbb{R} \setminus \{0\}$

$A \longmapsto \det A$.

Ex: $(\mathbb{C}, +) \longrightarrow (\mathbb{C}^\times, \times)$

$a \longmapsto \exp(2\pi\sqrt{-1}\, a)$

Thm: $\ker \varphi = \varphi^{-1}(1)$ is normal subgroup.

Def: An isomorphism is a bijective group homomorphism.

Equivalence relation:

$\sim$ is certain subset of $S \times S$. such that.

(write $a \sim b$ if $(a,b) \in \sim$)

- Transitive
- symmetric
- reflexive.

Partition: S = union of disjoint subsets

Equivalence relation $\Leftrightarrow$ Partition.

$C_a = \{ b \in S \mid a \sim b \}$ then $C_a = C_b$ or $C_a \cap C_b = \phi$.

$$S = \bigsqcup_{a \in S} C_a.$$

$$\bar{S} = \{ C_a \mid a \in S \}$$

Surjective map: $\pi : S \to \bar{S}$

Ex: $S = GL(n)$. $a \sim b$ if $\det a = \det b$.

Ex: $H \subset G$ subgroup

$a \sim b$ if $a = bh$ for some $h \in H$.

Coset: A left coset $aH = \{ ah \mid h \in H \}$.