Equivalence relation:

$\sim$ is certain subset of $S \times S$. such that.

(write $a \sim b$ if $(a,b) \in \sim$)

- Transitive
- symmetric
- reflexive.

Partition: S = union of disjoint subsets

Equivalence relation $\Leftrightarrow$ Partition.

$C_a = \{ b \in S \mid a \sim b \}$ then $C_a = C_b$ or $C_a \cap C_b = \phi$.

$$S = \bigsqcup_{a \in S} C_a.$$

$$\bar{S} = \{ C_a \mid a \in S \}$$

Surjective map: $\pi : S \to \bar{S}$

Ex: $S = GL(n)$. $a \sim b$ if $\det a = \det b$.

Ex: $H \subset G$ subgroup

$a \sim b$ if $a = bh$ for some $h \in H$.

Coset: A left coset $aH = \{ ah \mid h \in H \}$.

G/H = set of cosets.

Lagrange's Thm: $|G| = |H| \cdot |G/H|$.
(2.8.9)

Quotient group.

Def and Thm: If $N \subset G$ is a normal subgroup, then G/N has a natural structure of group, such that $G \to G/N$ is a group homomorphism.

Pf: Define $aN \cdot bN = (ab)N$.

(Need to check this well-defined).

If $aN = a'N$, $bN = b'N$, then

$abN = a'b'N$.

$a' = ah_1$, $b' = bh_2$.

$$a'b' = ah_1bh_2 = ab\underbrace{(b^{-1}h_1b)}_{\in H}\underbrace{h_2}.$$

(First isomorphism Thm)

If $\varphi: G \to G'$ is surjective homo with kernel N. then $\exists!$ isomorphism $\bar{\varphi}: G/N \to G'$, s.t.

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[dr, "\pi"'] & G' \\ & G/N \arrow[u, "\bar{\varphi}"'] \end{tikzcd}$$

Ex:
$$\mathbb{R} \to U(1) = \{ z \in \mathbb{C}^\times \mid |z| = 1 \}.$$
$$x \mapsto \exp(2\pi\sqrt{-1}\, x)$$

$$\mathbb{R}/\mathbb{Z} \cong S^1 \text{ (circle)}.$$

Ex: Cyclic groups $\mathbb{Z}$. $\mathbb{Z}/n\mathbb{Z} = C_n$.

Product group:

Defn: If G and G' are two groups, there is a natural group structure on its product $G \times G'$, defined by
$$(a, a') \cdot (b, b') = (ab, a'b')$$

Ex: $C_2 \times C_3 \cong C_6$.

Prop (2.11.4) let $H, K \subset G$ be subgroups.

$$f: H \times K \to G$$
$$(h, k) \mapsto hk.$$

is an group isomorphism if and only if $H \cap K = \{1\}$, $HK = G$, H, K are normal subgroups of G. and $hk = kh$ for $(h, k) \in H \times K$.