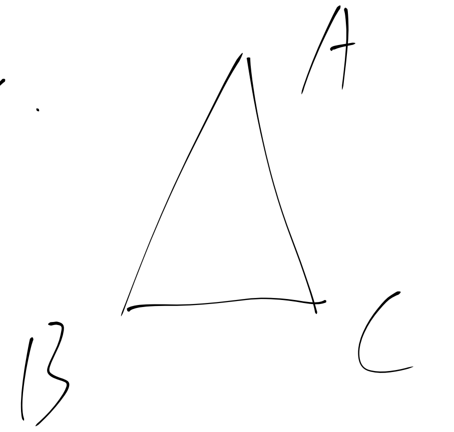
Symmetry

Ex. 

Symmetry of equilateral triangle. $\cong S_3$.

$\{1\}$, rotation by $120°$, rotation by $240°$.

reflections fixing A, or B, or C.

Ex. Symmetry of n elements $\cong S_n$.

Ex. Symmetry of vector space $\mathbb{R}^n$ $\cong GL(n)$.

Group operations (actions)

Defn: An operation of a group G on a set S is a map $G \times S \to S$ satisfying
$(g, s) \mapsto g.s$.

a) $1 \cdot s = s$

b) $g_1(g_2 \cdot s) = (g_1 g_2) \cdot s$.

Ex: $G = S_n$. $S = \{1, 2 \cdots n\}$.

$g \cdot k = g(k)$.

Left multiplication: $g \in G$, induces a <u>bijection</u>:

$$m_g : S \to S.$$
$$s \mapsto g \cdot s.$$

Why m_g is a bijection

$$(m_{g^{-1}} \circ m_g) = m_{g^{-1}g} = m_1 = id.$$

Another interpretation of Group operation.

Let $\text{Bijection}(S) = \{ f : S \to S \mid f \text{ is a bijection} \}$.

With the natural group structure by composition.

Then a group operation G on S is equivalent to a morphism: $G \to \text{Bijection}(S)$.

$$g \mapsto m_g.$$

More group actions.

$G \to S_n$. as a subgroup of S_n

Ex: G on G itself.

① Left multiplication $g \cdot s = g \cdot s$

(Cayley's Thm) $|G| = n$. then

② right multiplication $g \cdot s = s \cdot g^{-1}$

③ conjugation. $g \cdot s = gsg^{-1}$.

Orbits. $G \circlearrowright S$ (G operates on S).

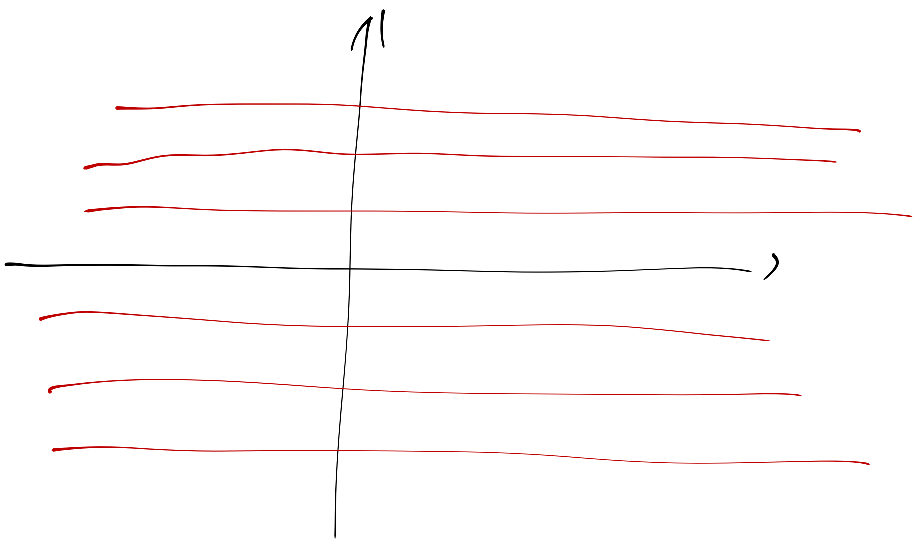
Defn: $s_1 \sim s_2$ if $g \cdot s_1 = s_2$ for some $g \in G$.

Equivalence classes under "$\sim$" are orbits of this action.

Ex: $\mathbb{R} \circlearrowright \mathbb{R}^2$.

$\mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$

$(a, (x, y)) \to (x+a, y)$



Ex: Conjugacy classes.

Orbits of conjugation.

Ex: left cosets. G/H.

orbits under right multiplication.

Defn: If S consists of one orbit. the operation of G is called transitive

$\boxed{\text{Decompose the action into actions on different orbits.}}$

Defn: Stabilizer $G_s = \{ g \in G \mid gs = s \}$

Prop: a) If $as = bs$, then $a^{-1}b \in G_s$

b). If $as = s'$. $G_{s'} = a G_s a^{-1}$

Operation on G/H.

Defn: $G \times G/H \to G/H$

$(g, aH) \mapsto gaH$.

Check: "well-defined":

If $aH = a'H$, then $gaH = ga'H$.

Prop: ① Transitive.

② Stabilizer. for $s = H$, is H.

for $s = aH$. $G_s = aHa^{-1}$

Prop: $G \curvearrowright S$, Let $s \in S$. $H = G_s$ stabilizer. O_s orbit

There is a bijection $f: G/H \to O_s$. Compatible with the group action. $aH \to as$.

$$\begin{array}{ccc} G \times G/H & \to & G/H \\ \downarrow{\scriptstyle id \times f} & & \downarrow{\scriptstyle f} \\ G \times O_s & \to & O_s \end{array}$$

$$f(g(aH)) = g \cdot f(aH).$$

Pf: "well-defined".

Check: $aH = a'H$, then $as = a's$.

f: injective. If $as = a's$, then $(a')^{-1}as = s$.

$h = (a')^{-1}a \in H$, $a = a' \cdot h$.

surjective. $s' \in O_s$, $s' = g \cdot s$.

so $f(gH) = s'$

compatible with G-operation.

$$f(g(aH)) = f(gaH) = gas$$

$$g \cdot f(aH) = g \cdot (as) = gas.$$

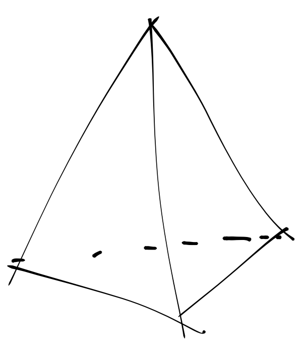
Counting formula.

Prop: $|G| = |G_s| \cdot |O_s|$

$$|S| = |O_1| + \cdots + |O_k|.$$

Ex: $S_n \curvearrowright \{1, \ldots, n\}$.

Ex: rotational symmetry of tetrahedron.



G

$$|G| = |G_s| \cdot |O_s| = 3 \cdot 4 = 12.$$