More examples of groups and group actions

① S_n acts on $\mathbb{R}^n$. (or $\mathbb{C}^n$). Conjugacy classes in S_n.

② Finite subgroups of $O(2)$, $SO(2)$.

Dihedral group D_n.

Cyclic group C_n.

③ Group action on set of subsets with fixed order.

①

S_n action on $\mathbb{R}^n$

$x \in S_n$. $\quad x : \{1 \dots n\} \mapsto \{1 \dots n\}$

$$i \longmapsto x_i .$$

$e_1 \dots e_n$ basis of $\mathbb{R}^n$.

$e_1 = (1, 0 \dots 0)^T \quad e_2 = (0, 1 \dots 0)^T$

$\vdots$

S_n acts on $e_1 \cdots e_n$. by

$$x(e_i) = e_{x(i)}.$$

then x extends to an action on

$$x\left(\sum a_i e_i\right) = \sum a_i\, x(e_i) = \sum a_i\, e_{x(i)}$$

So we have a homomorphism

$$\rho : S_n \to GL(n)$$

$$x \longmapsto$$

each row has exactly one "1".

column has exactly one "1"

$$\rho(xy) = \left[\quad\right]\left[\quad\right]$$

Determinant $\det : GL(n, \mathbb{R}) \to \mathbb{R}^\times$

Restriction to S_n.

$$S_n \to GL(n, \mathbb{R}) \to \mathbb{R}^\times$$

$$\text{sign} : S_n \to \mathbb{R}^\times$$

Question : (a) What is the image?

(b) what is the kernel?

(a) $\mathrm{Im}(\text{sign}) = \{\pm 1\}$.

(b) $\ker(\text{sign}) = A_n$. (even permutations)

A_n is a <u>index 2</u> normal subgroup of S_n

Pf: (a) $x^N = 1$ for $N = n!$

$(\text{sign}(x))^N = 1$.

so $\text{sign}(x) = \pm 1$.

Take $x(1) = 2$
$x(2) = 1$
$x(i) = i \quad i \geq 3.$

$\text{sign}(x) = -1$

More structures on S_n.

Defn: cycle $x = (i_1 \cdots i_k)$ $\quad i_1 \cdots i_k$ distinct.

$x(i_1) = i_2, \quad x(i_2) = i_3, \ldots$

$x(i_k) = i_1. \quad x(j) = j \text{ if } j \notin \{i_1 \cdots i_k\}$

Prop: If $x = (i_1 \cdots i_k)$ $\quad$ k is the length of x.
$y = (j_1 \cdots j_l)$
$\{i_1 \cdots i_k\} \cap \{j_1 \cdots j_l\} = \phi.$

then $xy = yx$. (disjoint cycles)

Thm (cycle decomposition).
Any $x \in S_n$ can be written as

$x = x_1 x_2 \cdots x_t$, x_i are disjoint cycles. $x_1 \cdots x_t$ is unique up to a permutation of index $1 \ldots t$.

Ex:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 2 & 5 & 3 & 4 & 1 \end{matrix}$$

$$x = (1\,8)(2\,7\,4)(3\,6)$$

Pf: Existence. $S = \{ i \mid x(i) \neq i \}$.

Induction on $|S|$ to prove $x = x_1 \cdots x_t$ and the union of elements appeared in x_i is S.

$$\{ i_1 \cdots i_k \} = \{ i \mid x(i) \neq i \}.$$

$$i_1,\ x(i_1) \ldots\ x^m(i_1).$$

$\exists\ n_1 \leq n_2$, s.t. $x^{n_1}(i_1) = x^{n_2}(i_1)$.

Take n_2 to be the first number such that

$$x^{n_1}(i_1) = x^{n_2}(i_1)$$

Then $x^{n_2 - n_1}(i_1) = i_1$.

so $n_1 = 0$, and $\underbrace{i_1, x(i_1) \dots x^{n_2-1}(i_1)}_{C}$

are distinct.

Let $X_1 = (i_1 \cdots x^{n_2-1}(i_1))$.

and $\tilde{x} = X_1^{-1} x$

$x(j) \notin \{i_1 \dots x^{n_2-1}(i_1)\}$ if $j \notin C$.

$\{\tilde{x}(i) \neq i\} = S - C$

Use induction assumption on $\tilde{x}$.

$\tilde{x} = x_2 \cdots x_t$

Uniqueness. If $x = x_1 \cdots x_t$

$= y_1 \cdots y_m$.

$\{x(i) \neq i\}$ is the union of elements appeared in

x_i, and also y_i.

so if $x_1(i_1) \neq i_1$, then i_1 must appear in some y_j.

recover y_j and x_1 use $i_1, x(i_1) \dots x^{n_2}(i_1)$

Each cycle decomposition corresponds to a partition of $n = k_1 + k_2 + \dots k_t + 1 \dots + 1$

$$\begin{aligned} 5 &= 2+3 \\ &= 3+2 \end{aligned}$$ same partition.

Thm: $x, y \in S_n$ are conjugate iff x, y corresponds to the same partition of n.

Pf: If $x = (i_1 \cdots i_k)$ is a cycle.

then $gxg^{-1} = (g(i_1) \cdots g(i_k))$.

If $x = x_1 \cdots x_t$.

then $gxg^{-1} = gx_1g^{-1}gx_2g^{-1} \cdots gx_tg^{-1}$

gx_ig^{-1} are disjoint cycles

So all the elements conjugate to x correspond to the same partition of n.

Conversely, if x, y correspond to the same partition of n. then we have cycle decompositions

$x = x_1 x_2 \cdots x_t$

$y = y_1 y_2 \cdots y_t$.

such that the length of x_i is the same as length of y_i,

assume $x_i = (a_1^i \cdots a_{k_i}^i)$

$y_i = (b_1^i \cdots b_{k_i}^i)$

and let $\{c_1 \cdots c_l\} = \{i \mid x(i) = i\}$.

$\{d_1 \cdots d_l\} = \{i \mid y(i) = i\}$.

Define $g(a_m^i) = b_m^i$

$g(c_i) = d_i$.

Then $gxg^{-1} = y$.

$\square$.

conclusion.

# of conjugacy classes = # of partitions of n.

Infinite group. $GL(2, \mathbb{R})$ acting on $\mathbb{R}^2$.

$$gv = \begin{bmatrix} \times & \times \\ \times & \times \end{bmatrix} v.$$

Put more structure on $\mathbb{R}^2$.

$$|v| = \sqrt{v_1^2 + v_2^2} \quad \text{or} \quad \langle v_1, v_2 \rangle = v_1^t v_2.$$

Defn ($O(2)$, orthogonal group)

The following are equivalent. (TFAE).

① $|gv| = |v|$ for all $v \in \mathbb{R}^2$