Define $g(a_m^i) = b_m^i$

$g(c_i) = d_i$.

Then $gxg^{-1} = y$.

$\square$.

conclusion.

# of conjugacy classes = # of partitions of n.

Infinite group. $GL(2, \mathbb{R})$ acting on $\mathbb{R}^2$.

$$gv = \begin{bmatrix} x & x \\ x & x \end{bmatrix} v.$$

Put more structure on $\mathbb{R}^2$.

$$|v| = \sqrt{v_1^2 + v_2^2} \quad \text{or} \quad \langle v, w \rangle = v^t w.$$

Defn ($O(2)$, orthogonal group)

The following are equivalent. (TFAE).

① $|gv| = |v|$ for all $v \in \mathbb{R}^2$

② $\langle gv, gw \rangle = \langle v, w \rangle$.

③ $g^t g = I$.

$|v|$ and $\langle \cdot \rangle$ are related by

$|v| = \sqrt{\langle v, v \rangle}$

$\langle v, w \rangle = \frac{1}{2}\left(|v+w|^2 - |v|^2 - |w|^2\right)$

(parallelogram law)

Structure of $O(2)$.

$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad g^t g = I$

$\Rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix}$.

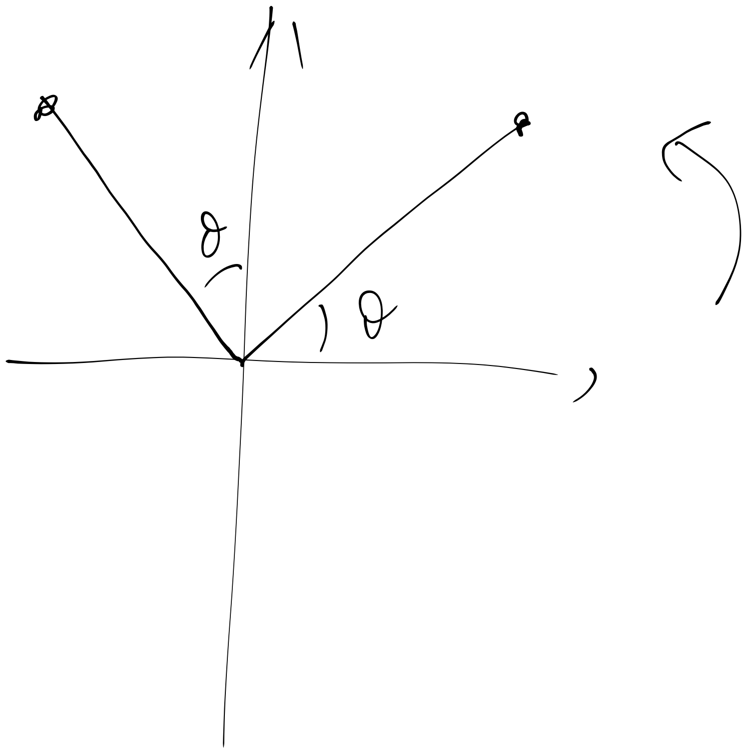
$a^2+c^2 = 1 = b^2+d^2$. $\quad ab+cd = 0$.

$a = \cos\theta, \quad b = -\sin\theta \qquad$ or $\qquad b = \sin\theta$

$c = \sin\theta \quad d = \cos\theta \qquad\qquad d = -\cos\theta$.

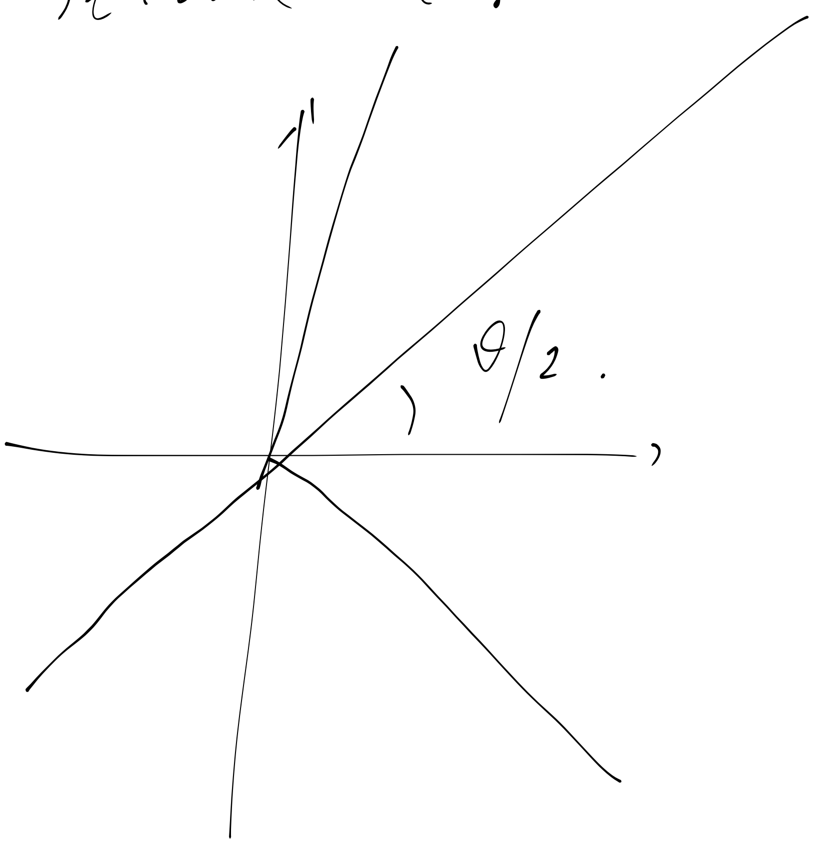
First case $\quad g = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

$$g e_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \qquad g(e_2) = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$



rotation.

second case.



reflection w.r.t $\theta/2$.

$$\mathrm{Det} : O(2) \to \{\pm 1\}.$$

$$\ker(\mathrm{Det}) = SO(2) = \left\{ \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\rho_\theta} \right\}.$$

Finite subgroups of $SO(2)$.

Thm: $G \subset SO(2)$ a finite subgroup.

then $G = \langle \rho_\theta \rangle$ $\theta = \frac{2\pi}{n}$.

and $G \cong C_n$.

Pf: (Euclidean division).

Let $\theta_1 = \min \{ \rho_\theta \in G \mid 0 < \theta < 2\pi \}$.

Then $\rho_{\theta_1} \in G$, $\langle \rho_{\theta_1} \rangle \subset G$.

If $G \not\subset \langle \rho_{\theta_1} \rangle$, then $\exists\, \rho_\theta \in G$.

s.t. $\rho_\theta \notin \langle \rho_{\theta_1} \rangle$

Let $\theta = m\theta_1 + r$. $m \in \mathbb{Z}_{\geq 0}$.

$0 \leq r < \theta$.

Since $\rho_\theta \notin \langle \rho_{\theta_1} \rangle$.

then $r > 0$,

so $\rho_\theta \cdot \rho_{-m\theta_1} = \rho_r \in G$

contradiction with definition of θ_1.

So $G = \langle \rho_{\theta_1} \rangle$ and for any

$\rho_\theta \in G$, $\theta = m \cdot \theta_1$.

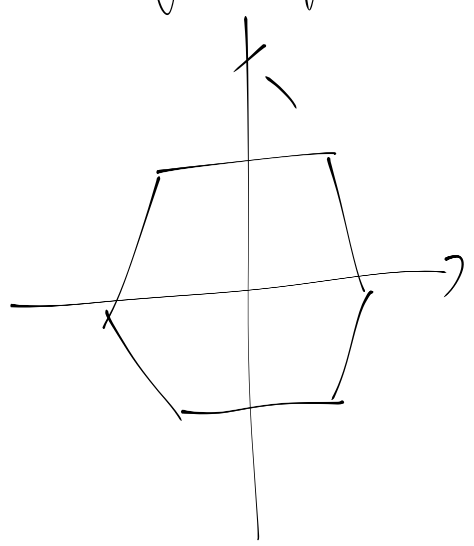
since ρ_{θ_1} has finite order.

$\theta_1 = \frac{2\pi}{n}$.

$\square$

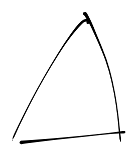
Finite subgroup of $O(2)$.

D_6



6 rotations
6 reflections.

D_3



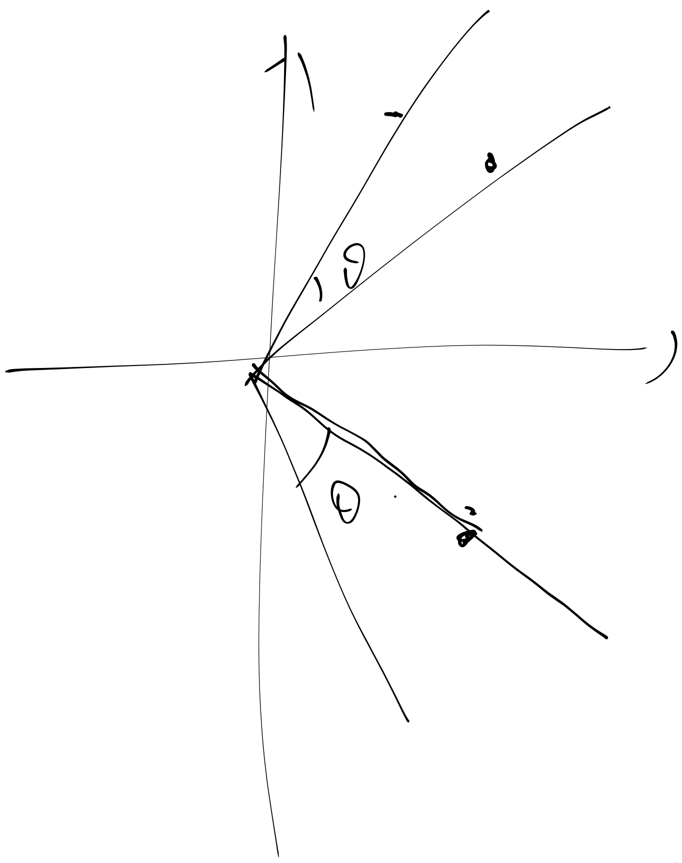
$D_3 \cong S_3$.

D_n symmetry of n-gon.

$x = \rho_\theta \qquad \theta = \frac{2\pi}{n}.$

$y = \text{reflection} \qquad y^2 = 1.$

$yxy^{-1} = x^{-1}.$



$$y = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$$

$$\text{or } y = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$$

$$y^2 = 1.$$

$$yxy^{-1} = x^{-1}.$$

elements in D_n: $1, x, \ldots, x^{n-1}, y, xy, \ldots, x^{n-1}y$

Thm (6.4.1) any finite subgroup G of O_2 is

(1) C_4

(2) D_n. generated by ρ_θ and reflection about a line l through the origin.

Pf: $G \subset SO(2)$. then case (1)

$G \not\subset SO(2)$. then $\exists\, y \in G$. $y \notin SO(2)$.

Assume $y = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$

$G \cap SO(2) = \langle \rho_\theta \rangle$.

Claim $D_n = \langle \rho_\theta, y \rangle = G$.

① $D_n \subset G$. obivious

② $D_n \supset G$. Any $g \in G$. $\forall g \notin SO(2) \cap G$.

then $gy \in G \cap SO(2)$.

so $g = (gy)y \in D_n$.

Conjugacy classes in D_n. $x = \rho_{\frac{2\pi}{n}}$, $y = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$

n even: $\{1\}$, $\{x, x^{-1}\}$, $\{x^2, x^{-2}\}$, $\cdots$ $\{x^{\frac{n}{2}}\}$.

$\{y, x^2y, x^4y, \cdots x^{n-2}y\}$

$\{xy, x^3y, \cdots x^{n-1}y\}$.

n odd: $\{1\}$, $\{x, x^{-1}\}$, $\{x^2, x^{-2}\}$, $\cdots$ $\{x^{\frac{n-1}{2}}, x^{\frac{n+1}{2}}\}$

$$\underbrace{\{y, x^2y, x^4y, \cdots x^{n+1}y = xy, x^3y, \cdots x^{n-2}y\}}_{\text{all the reflections}}.$$

Pf: Use the equalities

$(x^ky)x(x^ky)^{-1} = x^k yxy x^{-k} = x^k x^{-1} y^2 x^{-k} = x^{-1}$.

$x^k y x^{-k} = x^{2k}y$. $\quad x^k(x^my)x^{-k} = x^{2k+m}y$.

$(x^ky)y(x^ky)^{-1} = x^{2k}y$. $\quad (x^ky)x^my(x^ky)^{-1} = x^{2k-m}y$.

$= x^{2(k-m)}x^my$.