Generate new group actions by existing group actions.

① restrict to subgroup H

② act on set of subsets of a fixed order

Sylow's Thm

Defn: Center of G.
$Z(G) = \{g \in G \mid gh = hg$ for all $h \in G\}$
is a normal subgroup of G.

Defn: p-group. $|G| = p^n$.

Prop: Center of a p-group is nontrivial.

Consider the conjugate action of G on G.

$p^n = |G| = |O_1| + |O_2| \cdots |O_k|$.

$O_1 = \{1\}$. $\exists\ O_i$, s.t. $|O_i| = 1$.

Thm (Fix point Thm) $G \curvearrowright S$,
$p \nmid |S|$. then there is an element in S

s such that $G_s = G$. (s is fixed by G)

Prop: $|G| = p^2$, then G is abelian.

Pf: $G/Z(G) \neq \{1\}$. then $|Z(G)| = p$.

$\exists\ g \notin Z(G)$

Consider $Z(g) = \{h \in G \mid hgh^{-1} = g\}$.

centralizer.

then $Z(G) \subset Z(g)$.

and $g \in Z(g)$.

so $|Z(g)| > p$. $|Z(g)| = p^2 = |G|$

so $g \in Z(G)$. contradiction.

$\square$

Corollary: $|G| = p^2$, then $G \cong C_p \times C_p$

or $\cong C_{p^2}$

Pf: order of element in $G \mid p^2$.

① maximal order $= p^2$

$G = \langle g \rangle$ with $\operatorname{ord} g = p^2$

② maximal order $= p$.

then $G \supset \langle k \rangle$.

$G / \langle k \rangle \cong C_p$.

Choose $h \in G$, $h \notin \langle k \rangle$.

then $\underset{H}{\underset{\parallel}{\langle h \rangle}} \cap \underset{K}{\underset{\parallel}{\langle k \rangle}} = \{1\}$.

H, K both normal subgroups.

$|HK| > p$ $|HK| = p^2$, $HK = G$.

$G \cong H \times K$

How about $|G| = p^3$.

$$\left\{ \begin{bmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{bmatrix} \right\} \quad x \in \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p.$$

subgroup of $GL(3, \mathbb{F}_p)$.

What is the center?

$$\begin{bmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{bmatrix} \begin{bmatrix} 1 & x' & z' \\ & 1 & y' \\ & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & x'+x & z'+xy'+z \\ & 1 & y'+y \\ & & 1 \end{bmatrix}$$

If $xy' \neq x'y$, then they don't commute.

So $\begin{bmatrix} 1 & 0 & z \\ & 1 & 0 \\ & & 1 \end{bmatrix}$ is the center.

Question: What are the possible G, such that $|G| = p^3$.

More familiar example $G = D_4$ $|G| = 8$. D_n is not abelian when $n \geq 3$.

Question: Is $D_4 \cong \left\{ \begin{bmatrix} 1 & x & y \\ & 1 & z \\ & & 1 \end{bmatrix} \middle| \, x, y, z \in \mathbb{F}_2 \right\}$?

Normaliser. $N(H) = \{ g \in G \mid gHg^{-1} = H \}$.

Counting formula: $|G| = |N(H)| \cdot$ (number of conjugate subgroups of H.)

Prop: a) H is a normal subgroup of N.

b). H is normal in G iff $G = N(H)$

c). $|H| \mid |N|$. $|N| \mid G$.

Example: $p = (12)(34) \in S_5$.

gpg^{-1} has $\binom{5}{2}\binom{3}{2}/2 = \frac{5\times 4}{2\times 1} \times \frac{3\times 2}{2\times 1}/2$

$= 15$.

$|N(\langle p \rangle)| = \frac{120}{15} = 8$

Defn: Sylow p-group $|G| = p^e \cdot m$. $p \nmid m$.

subgroup $H \subset G$ such that $|H| = p^e$ is called Sylow p-group.

$|G/H| = [G:H]$ = index of H in G.