Normalizer. $N(H) = \{ g \in G \mid gHg^{-1} = H \}$.

Counting formula: $|G| = |N(H)| \cdot |\text{number of conjugate subgroups of } H.|$

Prop: a) H is a normal subgroup of N.

b). H is normal in G iff $G = N(H)$

c). $|H| \mid |N|$. $|N| \mid |G|$.

Example: $p = (12)(34) \in S_5$.

gpg^{-1} has $\binom{5}{2}\binom{3}{2}/2 = \frac{5\times 4}{2\times 1} \times \frac{3\times 2}{2\times 1}/2$

$= 15$.

$$|N(\langle p\rangle)| = \frac{120}{15} = 8$$

$\begin{pmatrix} 1&2&3&4&5 \\ 4&3&2&1&5 \end{pmatrix}$

$\begin{pmatrix} 1&2&3&4&5 \\ 4&3&1&2&5 \end{pmatrix}$

$\begin{pmatrix} 1&2&3&4&5 \\ 3&4&2&1&5 \end{pmatrix}$

$\begin{pmatrix} 1&2&3&4&5 \\ 3&4&1&2 \end{pmatrix}$

$N(\langle p\rangle) = \{ 1, \begin{pmatrix} 1&2&3&4&5 \\ 2&1&3&4&5 \end{pmatrix}, \begin{pmatrix} 1&2&3&4&5 \\ 1&2&4&3 \end{pmatrix}, \begin{pmatrix} 1&2&3&4&5 \\ 2&1&4&3&5 \end{pmatrix} \}$

Defn: Sylow p-group $|G| = p^e \cdot m$. $p \nmid m$.

subgroup $H \subset G$ such that $|H| = p^e$ is called Sylow p-group.

$|G/H| = [G:H] =$ index of H in G.

1st Sylow thm: (Existence).

If $p \mid |G|$, then G contains a sylow p-group.

2nd : (conjugate)

① The Sylow p-groups are conjugate.

② H subgroup that is a p-group is contained in a Sylow p-group.

3rd. $|G| = p^e m$. s = number of sylow p-groups.

$s \equiv 1 \mod p$. $s \mid m$.

Application: $|G| = 15$. then $G \cong C_{15}$.

H sylow 3-group $H \cong C_3 = \langle h \rangle$

K Sylow 5-group $K \cong C_5 = \langle k \rangle$.

H, K normal subgroups $HK = G$.

$H \cap K = \{1\}$, so $G \cong H \times K$.

$|G| = 6$, H sylow-3 group

K sylow-2 group.

H normal subgroup.

K normal or K_1, K_2, K_3 3-sylow group.

$G \curvearrowright \{[K_1], [K_2], [K_3]\}$ by conjugation.

$\rho: G \to S_3$. $\ker\rho = \{1\}$.

Pf of Sylow's Thms:

Lemma 1: U subset of G, $\mathrm{Stab}([U])$ of $[U]$ for the operation of left multiplication by G on the set of its subsets divides both $|U|$ an $|G|$.

Pf: $H = G_{[U]}$ then

$$U = \bigsqcup_{g \in U} Hg \quad \text{so } |H| \mid |U|.$$

Lemma 2: $|$Set of subsets with order $p^e| = N$.

$|S| = p^e m$. $p \nmid m$.

$p \nmid N$.

Pf: $N = \binom{n}{p^e} = \frac{n(n-1)\cdots(n-p^e+1)}{p^e(p^e-1)\cdots 1}$.

$k = p^l k_0$. $p \nmid k_0$. define $v(k) = l$.

For $1 \le k \le p^e - 1$. $v(k) < e$.

$v(p^e - k) = v(k)$. $\left(v(m_1 + m_2) = \min\{v(m_1), v(m_2)\} \right)$

$v(p^e m - k) = v(k)$.

$v(m_1 m_2) = v(m_1) \cdot v(m_2)$.

So $v(N) = v(n) - v(p^e) + v(n-1) - v(p^e - 1) \cdots$

$= 0$.

Pf of 1st Sylow's Thm:

Consider $S = \{ U \subset G \mid |U| = p^e \}$.

$|S| = N = \binom{p^e m}{p^e} \not\equiv 0 \pmod p$

$N = |O_1| + |O_2| \cdots |O_k| \not\equiv 0 \pmod p$.

$p^e m = |G| = |O_i| \cdot |G_i|$. G_i = stabilizer of $[U_i] \in O_i$.

$\exists i$, s.t. $p^e \mid |G_i|$.

Lemma1 $\Rightarrow$ $|G_i| \mid U_i$.

so $|G_i| = p^e$

2nd Sylow's Thm: K p-subgroup. H Sylow p-subgroup.

Consider the action of K on G/H.

K fix some aH. by fixed point theorem (proved last time)

then $K \subset aHa^{-1}$

3rd Sylow's Thm: G action on

$S = \{$ Sylow p-subgroups $\}$.

is transitive.

$|S| \cdot |N(H)| = |G|$.

$H \subset N(H)$. so $|S| \mid m$.

Restrict to H, splits into orbits.

$$|S| = |O_1| + |O_2| + \cdots |O_k|$$

$$\{[H]\} = O_1.$$

$$|O_k| \,\Big|\, |H| = p^e.$$

$|O_i| = 1$ means $O_i = \{[K]\}$.

and $gKg^{-1} = K$ for all $g \in H$.

$H \subset N(K)$.

Both H, K are sylow p-subgroups of $N(K)$.

So H, K are conjugate in $N(K)$.

So $H = K$. because K is normal subgroup of $N(K)$