02/11.

More applications of Sylow's Thms and Semi-direct product.

Classify Finite groups G of order 21.

# of 7-Sylow subgroup is 1.

# of 3-Sylow subgroup is 1 or 7.

Case 1. H unique Sylow 7-group.

H normal subgroup. $H \trianglelefteq G$.

K unique Sylow 3-group

$H \cap K = \{1\}$. $HK \cong H \times K$.

$HK = G$.

$G \cong C_3 \times C_7 \cong C_{21}$

Case 2. $K_1 \cdots \cdots K_7$ Sylow 3-groups

Let $K = K_1$

$$G \not\cong H \times K.$$

H normal subgroup $\Rightarrow$ $HK = KH$ subgroup (Homework 2, problem 3)

$H \cap K = \{1\}$.

$H \times K \to G$. (Note not a morphism)

$(h, k) \mapsto hk$.

Injective because $h_1 k_1 = h_2 k_2$.

$\Rightarrow$ $h_2^{-1} h_1 = k_2 k_1^{-1} \in H \cap K$.

Bijective because of the order $|H \times K| = |G|$.

Every element in G has a unique form

hk, $h \in H$, $k \in K$.

How to find the product structure?

$$hk\,h'k' = h(kh'k^{-1})kk'$$

Need to determine $kh'k^{-1}$

$$\varphi : K \longrightarrow \mathrm{Aut}(H).$$
$$k \longmapsto \varphi(k) : h \mapsto khk^{-1}$$

φ is a group morphism

$H = \langle x \rangle . \quad x^7 = 1.$

$K = \langle y \rangle \quad y^3 = 1.$

$y x y^{-1} = x^{i}?$

$\mathrm{Aut}(C_7) \cong (\mathbb{Z}/7\mathbb{Z})^{\times} \cong \mathbb{Z}/6\mathbb{Z}$

$$yxy^{-1} = x^j, \qquad y^2 x y^{-2} = y x^j y^{-1}$$

$$= (x^j)^j = x^{j^2}$$

$$y^3 x y^{-3} = x^{j^3} = 1.$$

so $j^3 \equiv 1 \pmod 7$.

$\bar{0}$ $\bar{1}$ $\bar{2}$ $\bar{3}$ $\bar{4}$ $\bar{5}$ $\bar{6}$

cube $\bar{0}$, $\bar{1}$, $\bar{1}$, $\bar{6}$, $\bar{1}$, $\bar{6}$, $\bar{6}$

$j = 2$ or 4. (choose y^2 instead of y makes $j = 2$).

so $yxy^{-1} = x^2$

Defn (outer semi direct product). H, K groups. $\varphi: K \to \mathrm{Aut}(H)$ is a homomorphism.

There is a group structure on $H \times K$ by

$$(h, k) \cdot (h', k') = (h \cdot \varphi(k) h', \; k k')$$

(Check this defines a group structure.)

It is denoted by $H \rtimes_{\varphi} K$.

Thm: If H is a normal subgroup of G, K is a subgroup of G, $H \cap K = \{1\}$, and $G = HK$, then G is isomorphic with the semi direct product $H \rtimes_{\varphi} K$ with

$$\varphi: K \to \mathrm{Aut}\, H$$
$$k \mapsto \varphi(k): h \mapsto k h k^{-1}.$$