Review for 1st midterm:

Defn: Groups, subgroups, normal subgroups.

cyclic group, homomorphism, isomorphism,

Quotient group, 1st isomorphism thm.

Group operation orbits, stabilizer.

Conjugation centralizer, normalizer.

Conjugacy classes.

Counting formula.

p-groups, $|G|=p$, $G \cong C_p$.

$|G|=p^2$, $G \cong C_p \times C_p$, C_{p^2}.

$|G|=p^3$ can be non abelian.

Sylow's Thms

Ex: $A_n \subset S_n$. D_n,

$SL(n) \subset GL(n)$

$SO(2) \subset O(2)$

finite subgroups in $O(2)$ and $SO(2)$

Classify G of order 12.

$|G| = 12. = 2^2 \times 3$.

$|\{\text{Sylow 2-groups}\}| = s = 1$ or 3.

$|\{\text{Sylow 3-groups}\}| = s' = 1$ or 4,

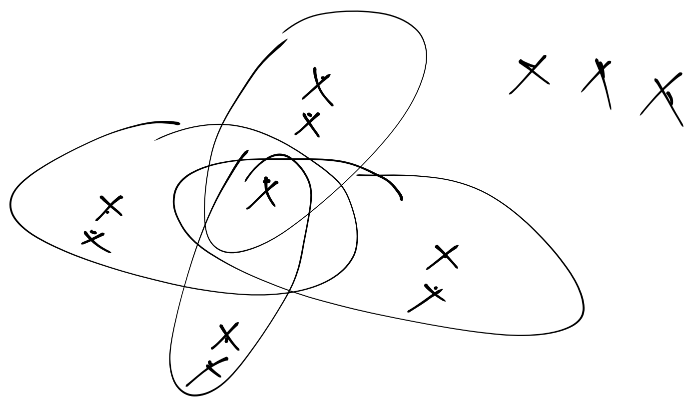
If $s = 3$, $s' = 4$,

there're 4 Sylow 3-groups.

K_1, K_2, K_3, K_4

$K_i \cap K_j = \{1\}$ for $i \neq j$ because they're cyclic.

so



Let H be Sylow-2group.

then $H \subset \{1\} \cup (K_1 \cup K_2 \cup K_3 \cup K_4)^c$

and $|H| = 4$

so H is unique.

Case 1 $H \lhd G$,

Case 2 $K \lhd G$.

1a $H \cong C_2 \times C_2$, $K = C_3 = \langle y \rangle$.

Let $H = \langle x_1, x_2 \rangle$ $x_1^2 = x_2^2 = 1$. $x_1 x_2 = x_2 x_1$

Let $f \in Aut(H)$

then $f\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}a_{11}\\a_{12}\end{pmatrix}$

$$f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix}$$

So $|\mathrm{Aut}(H)| = (2^2-1)(2^2-2) = 6$

$$f = \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right.$$
$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\varphi: K \to \mathrm{Aut}(H)$$
$$y \mapsto \varphi(y) = f.$$

$$f^3 = 1. \Rightarrow f = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$
$$\text{or } \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

up to a choice of generators for H (or K).

we can assume $f = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

so $G = \langle x_1, x_2, y \rangle$.

$$x_1^2 = x_2^2 = 1. \quad x_1 x_2 = x_2 x_1$$

$$y x_1 y^{-1} = x_2, \quad y x_2 y^{-1} = x_1 x_2.$$

(actually isomorphic to A_4) or $G \cong C_2 \times C_2 \times C_3$.

1b. $H = C_4, \quad K = C_3$.

$$\mathrm{Aut}(H) = (\mathbb{Z}/4\mathbb{Z})^\times \cong (\mathbb{Z}/2\mathbb{Z}).$$

no nontrivial homomorphism from C_3 to

$\mathrm{Aut}(H)$

so $G \cong C_3 \times C_4$.

2a: $H = C_2 \times C_2, \quad \mathrm{Aut}(C_3) = (\mathbb{Z}/3\mathbb{Z})^\times \cong (\mathbb{Z}/2\mathbb{Z})$

$= \langle x_1, x_2 \rangle. \quad x_1^2 = x_2^2 = 1, \; x_1 x_2 = x_2 x_1$

$\varphi: H \to \mathrm{Aut}(C_3)$.

$x_1 y x_1^{-1} = y^{j_1}$. $x_2 y x_2^{-1} = y^{j_2}$.

$j_1^2 \equiv j_2^2 \equiv 1 \mod 3$,

So $(j_1, j_2) = (1, 1)$ $G \cong C_3 \times C_2 \times C_2$

$(j_1, j_2) = (1, 2)$ or $(2, 1)$.

$x_1 y x_1^{-1} = y$, $x_2 y x_2^{-1} = y^2$

In this case $G \cong D_6$.

$(j_1, j_2) = (2, 2)$. choose $x_1^{-1} x_2$, x_2 as generator for H. reduce to

$(j_1, j_2) = (1, 2)$

2b. $H = C_4$, $\mathrm{Aut}(K) = (\mathbb{Z}/3\mathbb{Z})^\times$

So $xyx^{-1} = y$ or y^2

If $xyx^{-1} = y$ then $G \cong C_4 \times C_3$

$xyx^{-1} = y^2$, then $G \cong C_3 \rtimes_\varphi C_4$.

$G = \langle x, y \rangle$, $x^4 = 1$, $y^3 = 1$, $xyx^{-1} = y^2$

In total, there are 5 isomorphism classes.