(Idempotent element) $e \in R$, $e^2 = e$.

Prop: a). $e' = 1-e$ is also idempotent.

b). eR is also a ring with identity e.

(Notice that eR is not a subring)

c). $R \cong eR \times e'R$

Pf: a) $(1-e)^2 = 1-2e+e^2 = 1-e$.

b). $\forall\ ea \in eR$.

$e\cdot(ea) = e^2 a = ea$

c). $R \longrightarrow eR \times e'R$.

$a \mapsto (ea, e'a)$

idea $e+e' = 1$.

$(e+e')a = ea + e'a$

bijection.

ring homomorphism.

Example of product ring and idempotent elements

Ex: $\mathbb{F}_2[x] / x^2+x. = R$

$0, 1, x, x+1.$

$x^2 = x$, $(x+1)^2 = x^2+2x+1$
$= (x^2+x)+x+1 = x+1$

So $\mathbb{F}_2[x] / (x^2+x) \cong \underbrace{Rx} \times R(x+1)$

Ex:

$\mathbb{Z}/6\mathbb{Z} \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$

Non Ex. $\mathbb{Z}/8\mathbb{Z} \ncong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$

(Chinese Remainder theorem). $I + J = R$.
Then, $IJ = I \cap J$,

$$R/IJ \cong (R/I) \times (R/J).$$

(Hint: $R \to (R/I) \times (R/J)$ is surjective.
$a \mapsto (a+I,\ a+J)$)

Maximal ideal.

$I \subsetneq R$ is a maximal ideal.

iff Any ideal $J \supset I$, $J = I$ or R.

Prop: $I \subseteq R$ is a maximal ideal iff R/I is a field

Pf: Use correspondence thm and the fact that any ring F is a field iff F has only two ideals (0) and F itself.

Example: $R = \mathbb{C}[x,y]$. (Find maximal ideals in R)

Define $\varphi: R \to \mathbb{C}$ $\quad (\alpha_1, \alpha_2) \in \mathbb{C}^2$

$x \mapsto \alpha_1$ $\quad$ a ring homo

$y \mapsto \alpha_2$.

then $\ker\varphi = (x-\alpha_1, y-\alpha_2)$ (Think why?)

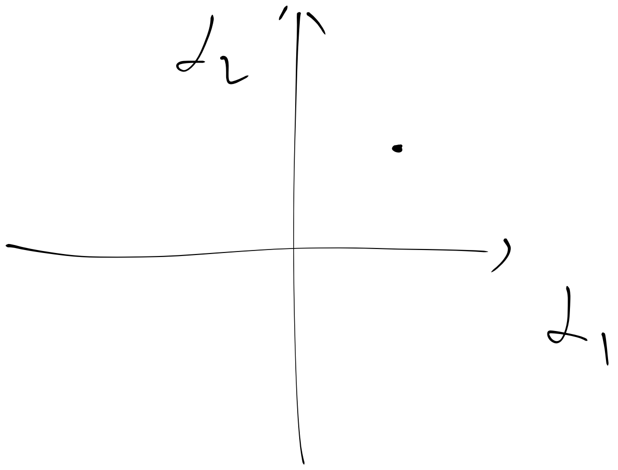
Since φ is a surjective map.

1st isomorphism Thm $\Rightarrow R/\ker\varphi \cong \mathbb{C}$.

So $(x-\alpha_1, y-\alpha_2)$ is a maximal ideal.

The converse is also true, this is the famous Hilbert's Nullstellensatz.

Thm: All the maximal ideals in $\mathbb{C}[x_1 \dots x_n]$ are of the form $(x_1-\alpha_1, x_2-\alpha_2, \dots, x_n-\alpha_n)$ for some $(\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{C}^n$

Picture 

maximal ideals in $\mathbb{C}[x,y]$

Ex: $\mathbb{C}[x,y]/(y-x^2) =: R$

$\{$maximal ideals in $R\}$ $\xleftrightarrow{1:1}$

$\{$maximal ideals in $\mathbb{C}[x,y]$ containing $(y-x^2)\}$

All the maximal ideals in R are in the form $(x-\alpha_1, y-\alpha_2)$ such that $\alpha_2 = (\alpha_1)^2$

Pf: If $(x-\alpha_1, y-\alpha_1) \supset (y-x^2)$

Then $\varphi: \mathbb{C}[x,y] \to \mathbb{C}$.

$$x \longmapsto \alpha_1$$
$$y \longmapsto \alpha_2$$

$(y - x^2) \subset \ker \varphi$.

then $\varphi(y - x^2) = 0 \Rightarrow \alpha_2 - \alpha_1^2 = 0$.

The converse is also true.