Some definitions to clarify:

① Integral domain (domain)

ring without zero divisors.

② Polynomial ring: $R[x]$

"constant" means $R \subset R[x]$

③ monic polynomial

$f(x) = x^n + a_{n-1}x^{n-1} + \cdots - a_0.$

leading coefficient $=1$.

④ Field F

the set of units is $F \setminus \{0\}$

$$\boxed{\begin{array}{l}\text{Criterion for maximal ideals.}\\ \quad I \subset R \text{ is an ideal in } R.\\ \quad I \text{ is maximal ideal iff } R/I \text{ is a field.}\end{array}}$$

Example 1: $R = \mathbb{Z}$.

All the ideals in $\mathbb{Z}$ are in the form of (n). $n \geq 0$. $n \in \mathbb{Z}$

① If n is a prime number.

then $\mathbb{Z}/(n) = \mathbb{Z}/n\mathbb{Z}$ is a field $\mathbb{F}_n$

(We proved this before)

so (n) is an maximal ideal.

A more direct approach from definition.

If $J \supset (n)$ is another ideal containing (n). We write $J = (m)$.

Then $(n) \subset (m)$. so $n = m \cdot k$.

Since n is a prime number, according to fundamental theorem of arithmetic

$m = \pm n$ or $m = \pm 1$.

If $m = \pm n$, then $(m) = (n)$

If $m = \pm 1$. then $(m) = \mathbb{Z}$

Another fact we usually use:

$(x) = R$ iff x is a unit.

① If x is a unit, $1 = x \cdot x^{-1} \in (x)$

$r = r \cdot 1 \in (x)$.

② If $(x) = R$, then $1 = x \cdot a$ for some a.

(2) If n is not a prime.

$n = m_1 m_2$, $\quad m_i \neq \pm 1$

so $\overline{m_1} \in \mathbb{Z}/n\mathbb{Z} \neq \bar{0}$

$\overline{m_2} \in \mathbb{Z}/n\mathbb{Z} \neq \bar{0}$

$\overline{m_1} \cdot \overline{m_2} = \bar{n} = \bar{0}$

so $\mathbb{Z}/n\mathbb{Z}$ has zero divisors.

$\mathbb{Z}/n\mathbb{Z}$ is not an integral domain.

hence not a field.

Example: $R = F[x]$. F is a field.

What are the maximal ideals in R?

All the ideals in R are in the form

$(f(x))$ $f(x)$ is a monic polynomial.

Def: $f(x)$ is irreducible polynomial in $F[x]$ iff

(F is a field)

① $f(x) \neq 0$ $f(x)$ is not a constant.

② If $f(x) = g(x) \cdot h(x)$. $g(x), h(x) \in F[x]$

then $g(x)$, or $h(x)$ must be constant.

Claim: $(f(x))$ is a maximal ideal iff $f(x)$ is irreducible.

Pf: "$\Leftarrow$" If $f(x)$ is irreducible.

Assume $J = (g(x)) \supset (f(x))$.

then $f(x) = g(x) \cdot h(x)$

① If $g(x)$ is constant.

then $g(x)$ is invertible.

$(g(x)) = F(x)$

② If $h(x)$ is constant.

$g(x) = (h(x))^{-1} \cdot f(x)$

$(g(x)) = (f(x))$

" $\Longrightarrow$ " If $(f(x))$ is a maximal ideal

Assume $f(x) = g(x) \cdot h(x)$

then $(g(x)) \supset (f(x))$

① $(g(x)) = F[x]$. then

$1 = g(x) \cdot m(x)$. $\deg g = 0$.

$g(x)$ is a constant

② $(g(x)) = (f(x))$. then

$g(x) = f(x) \cdot H(x)$.

so $f(x) = f(x) \cdot H(x) \cdot h(x)$.

$\deg H = \deg h = 0$

$h(x)$ is a constant

Ex: $\mathbb{F}_2[x] / (x^2+x+1)$

$f(x) = x^2+x+1$ is irreducible.

because if $f(x) = g(x) h(x)$

and $\deg g \neq 0$. $\deg h \neq 0$.

then $\deg g = \deg h = 1$.

$g(x) = x$ or $x+1$

If $g(x) = x$, $f(0) = g(0) h(0) = 0 \cdot h(0) = 0$

but $f(0) = 1$

If $g(x) = x+1$, $f(1) = g(1) \cdot h(1) = 0 \cdot h(1) = 0$

but $f(1)=1$

so $f(x)$ is irreducible and

$\mathbb{F}_2[x]/(x^2+x+1)$ is a field.

Example (revisited).

$R = \mathbb{C}[x,y]$. Construct maximal ideal.

$$\varphi_{\alpha_1,\alpha_2} : \mathbb{C}[x,y] \to \mathbb{C}$$
$$f(x,y) \mapsto f(\alpha_1, \alpha_2)$$

surjective.

$\ker \varphi_{\alpha_1,\alpha_2} = (x-\alpha_1, y-\alpha_2)$.

(Why ?).

$\ker \varphi_{\alpha_1,\alpha_2} \supset (x-\alpha_1, y-\alpha_2)$. (use definition).

Look at the special case. $\alpha_1 = \alpha_2 = 0$

$$f(x,y) = a_{00} + a_{10}x + a_{01}y + a_{11}xy + a_{20}x^2 + a_{02}y^2 + \cdots$$

$$\varphi_{0,0}(f(x,y)) = f(0,0) = a_{00}$$

$f \in \ker \varphi_{0,0} \iff f(0,0) = 0 \iff$

$$f \in (x, y)$$

For different (α_1, α_2).

$(x-\alpha_1, y-\alpha_2)$ is different.

i.e. If $(\alpha_1, \alpha_2) \neq (\beta_1, \beta_2)$.

then $(x-\alpha_1, y-\alpha_2) \neq (x-\beta_1, y-\beta_2)$.

Pf: assume $\alpha_1 \neq \beta_1$, and

$(x-\alpha_1, y-\alpha_2) = (x-\beta_1, y-\beta_2) = I$.

then $(x-\alpha_1)-(x-\beta_1)=\beta_1-\alpha_1 \neq 0 \in I$.

$\beta_1-\alpha_1$ is a unit, so $I=\mathbb{C}[x,y]$

contradiction!

Hilbert's Nullstellensatz says

There is a one-to-one correspondence:

$$\mathbb{C}^2 \longleftrightarrow \{\text{maximal ideals in } \mathbb{C}[x,y]\}$$

$$(\alpha_1, \alpha_2) \longmapsto (x-\alpha_1, y-\alpha_2)$$

(we proved "well-defined", "injective".
Hilbert proved surjectivity)

Corollary: consider $R = \mathbb{C}[x,y]/V$.

$$V = (f_1, f_2, \cdots f_n)$$

then there is a bijection

$$\left\{ (\alpha_1, \alpha_2) \;\middle|\; \begin{array}{c} f_1(\alpha_1, \alpha_2) = 0 \\ \vdots \\ f_n(\alpha_1, \alpha_2) = 0 \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{maximal} \\ \text{ideals} \\ \text{in } R \end{array} \right\}$$

$$(\alpha_1, \alpha_2) \longmapsto (x - \alpha_1, y - \alpha_2)$$

Pf: Use correspondence theorem:

$$\{ \text{maximal ideals in } R \}$$

$$\xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{maximal ideals in } \mathbb{C}[x, y] \\ \boxed{\text{containing } V} \end{array} \right\}$$

How to check containing V?

$f_i(x, y) \in (x - \alpha_1, y - \alpha_2) \iff f_i \in \ker \varphi_{\alpha_1, \alpha_2}$

$\varphi_{\alpha_1, \alpha_2}(f_i(x, y)) = 0 \iff f_i(\alpha_1, \alpha_2) = 0$

So we have the correspondence above