So $(a_j) = (a_{j+1}) = \cdots$

Non UFD.

$\mathbb{Z}[\sqrt{-5}]$.

$6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$.

$2.\ 3.\ 1+\sqrt{-5}.\ 1-\sqrt{-5}$ are all irreducible.

If $(a+b\sqrt{-5})(c+d\sqrt{-5}) = 2$.

$$\begin{cases} ac-5bd = 2 \\ ad+bc = 0 \end{cases}$$ hard to solve.

Instead

$|a+b\sqrt{-5}|^2\,|c+d\sqrt{-5}|^2 = 4$.

$\underline{(a^2+5b^2)(c^2+5d^2) = 4}$.

$\Rightarrow\ a^2+5b^2 = 1.\ 2.\ 4.$ has to be 1.

$$(a+b\sqrt{-5})(c+d\sqrt{-5}) = 1+\sqrt{-5}.$$

$$(a^2+5b^2)(c^2+5d^2) = 6.$$

$$a^2+5b^2 = 1, \cancel{2}. 3. 6$$

$$a^2+5b^2 = 1. \ 6.$$

The units in $\mathbb{Z}[\sqrt{-5}]$ are ± 1.

(Similar method by taking $1 \cdot 1$)

Application.

GCD: $d \mid a$. $d \mid b$.

if $e \mid a$. $e \mid b$. then $e \mid d$.

$a = p_1 \cdots p_m$

$b = q_1 \cdots q_n$.

compare $p_1 \cdots p_m$

$q_1 \cdots q_n$.

a, b coprime if $GCD(a,b) = 1$.

Fermat Last thm:

$$x^n + y^n = z^n \quad xyz \neq 0$$

has no integer solutions.

Polynomial version:

$$f^n + g^n = h^n$$

has no solution in $\mathbb{C}[t]$ such that $g.c.d(f, g) = 1$, $\deg f \geq 1$.

Pf: Assume there is solution (f, g, h).

Choose (f, g, h) such that $\deg f + \deg g + \deg h$ achieves minimal

$$f^n = \prod_{k=0}^{n-1} (h - \xi_k g)$$

$$\xi_k = e^{\frac{2\pi i}{n} \cdot k}$$

$g.c.d\,(h, g) = 1 \Rightarrow$

$g.c.d\,(h - \xi_k g,\ h - \xi_l g) = 1$

for $k \neq l$

(why! h, g can be represented by $h - \xi_k g$ and $h - \xi_l g$.

Let $H = h - \xi_k g$

$G = h - \xi_l g$

$$h = \frac{\xi_l H - \xi_k G}{\xi_l - \xi_k}.$$

$$g = \frac{H - G}{\xi_l - \xi_k}.$$)

From UFD.

$$h - \xi_i g = (x_i(t))^n.$$

$\Rightarrow$

$$h - g = x(t)^n$$
$$h - \xi_1 g = y(t)^n$$
$$h - \xi_2 g = z(t)^n.$$

$\Rightarrow$ solve h, g

$\Rightarrow$ after absorbing constants to the n-th power.

$$x(t)^n + y(t)^n = z(t)^n.$$

with lower degrees.