Math 371 **Name:** ________________________
Spring 2020
Midterm 1
2/20/2020
Time Limit: 80 Minutes ID ________________________

"My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this"

Signature ________________________

This exam contains 9 pages (including this cover page) and 6 questions.
Total of points is 70.

- Check your exam to make sure all 9 pages are present.
- You may use writing implements on both sides of a sheet of 8"x11" paper.
- NO CALCULATORS.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	20	
Total:	70	

1. (10 points) Define a symmetric bilinear form on $\mathbb{R}^3$ by $\langle X, Y \rangle = X^T A Y$ where $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Find a basis v_1, v_2, v_3 such that $\langle v_i, v_j \rangle = 0$ for all $i \neq j$.

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

2. (10 points) Let $i = \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}$ and $j = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ be two elements in $SU(2)$. Determine whether they are in the same conjugacy class. If they are, find $P \in SU(2)$ such that $PiP^{-1} = j$. If not, state the reason.

Trace i = Trace $j = 0$.

So i, j are in the same conjugacy class.

$Pi = jP \qquad P = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}$

$j \cdot v_1 = \sqrt{-1}\, v_1. \qquad v_1 = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \sqrt{-1} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \right) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-\sqrt{-1}\, a + b = 0 \Rightarrow b = \sqrt{-1}\, a.$

$|a|^2 + |b|^2 = 1.$ So $\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \sqrt{-1} \end{pmatrix}$

$P = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}$ so $P = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{-1}}{\sqrt{2}} \\ \frac{\sqrt{-1}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$

3. (10 points) (a) Find an injective group homomorphism from $SO(2)$ to $SO(3)$.

(b) Find an injective group homomorphism from $O(2)$ to $SO(3)$.

a) $f: SO(2) \to SO(3)$

$$A \mapsto f(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & & \\ 0 & A & \end{bmatrix}$$

b). $\varphi: O(2) \to SO(3)$

$$A \mapsto \varphi(A) = \begin{bmatrix} \det A & 0 & 0 \\ 0 & & \\ 0 & A & \end{bmatrix}$$

4. (10 points) Let $e_1 \cdots e_n$ be the standard basis of $\mathbb{C}^n$. Define the linear operation ρ of permutation group S_n on $\mathbb{C}^n$ by $\sigma \cdot e_i = e_{\sigma(i)}$. Denote by χ_ρ the corresponding character. Prove that $\chi_\rho(\sigma)$ is equal to the number of elements fixed by σ, i.e.

$$\chi_\rho(\sigma) = \text{the number of elements in } \{i \in \{1, 2, \cdots, n\} | \sigma(i) = i\}.$$

Choose basis $\beta = \{e_1 \dots e_n\}$.

$$R_\beta(\sigma) = \begin{bmatrix} \sigma_{11} & \sigma_{12} & & \\ \sigma_{21} & \vdots & \cdots & \\ \vdots & \vdots & & \\ \sigma_{n1} & \sigma_{n2} & & \sigma_{nn} \end{bmatrix}$$

$$\sigma(e_i) = \sum_{k=1}^{n} \sigma_{ki} \cdot e_k = e_{\sigma(i)}$$

so $\sigma_{ki} = 0$ if $k = \sigma(i)$

$\sigma_{ki} = 1$ if $k \neq \sigma(i)$.

$$\chi_\rho(\sigma) = \sum_{i=1}^{n} \sigma_{ii} = \underbrace{1 + 1 + \cdots + 1}_{\substack{\text{number of } i \text{ such} \\ \text{that } i = \sigma(i).}}$$

5. (10 points) Prove that any rotation in $SO(2)$ can be written as the product of two reflections in $O(2)$.

$$y_{\frac{\theta}{2}} = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$ is a reflection.

$$y_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$y_{\frac{\theta}{2}} y_0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

is a rotation by angle θ.

θ can be any real number.

6. (20 points) Let W be the space of real trace-zero 2×2 matrices $W = \{A \in M_{2\times 2}(\mathbb{R}) | trace(A) = 0\}$. W has a basis $\mathbf{B} = (w_1, w_2, w_3)$, where

$$w_1 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, w_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, w_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

   (a) Show that the symmetric bilinear form defined by $\langle A, A' \rangle = \text{trace}\,(AA')$ has signature $(2, 1)$. (Hint: use basis $\mathbf{B}$)
   (b) Prove that $P \star A = PAP^{-1}$ defines a linear group operation of $SL(2, \mathbb{R})$ on the space W.
   (c) Use this operation to define a group homomorphism $\varphi : SL(2, \mathbb{R}) \to O_{2,1}$.
   (d) Prove the kernel of this homomorphism is $\{\pm I\}$.

a) $\langle w_1, w_2 \rangle = tr\left(\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$

$= tr \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = 0$

$\langle w_1, w_3 \rangle = tr\left(\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)$

$= tr\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = 0$

$\langle w_2, w_3 \rangle = tr\left(\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\right)$

$= tr\left(\begin{bmatrix} -1 & \\ & 1 \end{bmatrix}\right) = 0.$

Draft 1:

If you use this page and want it looked at, then you must indicate so on the page with the original problem on it. Make sure you label your work with the corresponding problem number.

$$\langle w_1, w_1 \rangle = \operatorname{tr}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = 2 > 0$$

$$\langle w_2, w_2 \rangle = \operatorname{tr}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = 2 > 0$$

$$\langle w_3, w_3 \rangle = \operatorname{tr}\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} = -2 < 0.$$

b) If $\operatorname{tr} A = 0$, then $\operatorname{tr}(PAP^{-1}) = 0$.

$$SL_2(\mathbb{R}) \times W \to W.$$

$$(P, A) \mapsto PAP^{-1}$$

is well-defined.

(1) $P_1 * (P_2 * A)$

$$= P_1 (P_2 A P_2^{-1}) P_1^{-1}$$

$$= (P_1 P_2) A (P_1 P_2)^{-1} = (P_1 P_2) * A.$$

Draft 2:

If you use this page and want it looked at, then you must indicate so on the page with the original problem on it. Make sure you label your work with the corresponding problem number.

(2) $I_2 * A = A$.

(3) $P*(A_1+A_2) = P(A_1+A_2)P^{-1}$

$= PA_1P^{-1} + PA_2P^{-1}$

$= P*A_1 + P*A_2$.

(4) $P*(cA) = P(cA)P^{-1}$

$= c\,PAP^{-1} = c\,P*A$.

c). $\langle P*A, P*A' \rangle$

$= tr(PAP^{-1} \cdot PA'P^{-1})$

$= tr(AA')$.

So $*$ defines a group homomorphism

Draft 1:

If you use this page and want it looked at, then you must indicate so on the page with the original problem on it. Make sure you label your work with the corresponding problem number.

d): $P \in \ker \varphi$

$\Leftrightarrow PAP^{-1} = A \quad \forall A \in W$.

$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad A = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then $b = c = 0$.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$\Rightarrow \quad a = d. \qquad \begin{bmatrix} c & a \end{bmatrix}$

Draft 2:

If you use this page and want it looked at, then you must indicate so on the page with the original problem on it. Make sure you label your work with the corresponding problem number.

$\det \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = 1 \Rightarrow \quad ad = 1.$

$\Rightarrow \quad a = d = \pm 1 \quad \Rightarrow \ker \varphi \subset \{\pm I\}.$

$(\pm I) A = A \cdot (\pm I)$

so $\ker \varphi = \{\pm I\}.$