Math 371 **Name:** ____________________
Spring 2020
Practice 1
2/20/2020
Time Limit: 80 Minutes ID ____________________

"My signature below certifies that I have complied with the University of Pennsylvania's Code of Academic Integrity in completing this"

Signature ____________________

This exam contains 9 pages (including this cover page) and 6 questions.
Total of points is 70.

- Check your exam to make sure all 9 pages are present.
- You may use writing implements on both sides of a sheet of 8"x11" paper.
- NO CALCULATORS.
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Good luck!

Grade Table (for teacher use only)

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	20	
Total:	70	

1. (10 points) Define a symmetric bilinear form on $\mathbb{R}^3$ by $\langle X, Y\rangle = X^T AY$ where $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. Find a basis v_1, v_2, v_3 such that $\langle v_i, v_j\rangle = 0$ for all $i \neq j$.

$V_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\langle V_1, V_1 \rangle = 2.$

$\langle V_1, V_2 \rangle = 1$

$\langle V_1, V_3 \rangle = 2$

$V_2' = V_2 - \dfrac{\langle V_1, V_2 \rangle}{\langle V_1, V_1 \rangle} V_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \dfrac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \quad V_2'' = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$V_3' = V_3 - \dfrac{\langle V_1, V_3 \rangle}{\langle V_1, V_1 \rangle} V_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\langle V_2'', V_3' \rangle = (-1) + (-1) + 1 = -1$

$V_3'' = V_3' - \dfrac{\langle V_3', V_2'' \rangle}{\langle V_2'', V_2'' \rangle} V_2'' = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \dfrac{-1}{-2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$, $\{V_1, V_2'', V_3''\}$ satisfies the requirement.

2. (10 points) Find an injective group homomorphism from $U(1)$ to $SU(2)$.

$$U(1) \to SU(2)$$

$$e^{i\theta} \mapsto \begin{bmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{bmatrix}$$

3. (10 points) Let $A \in U(n)$ be a unitary matrix. Let v_1 and v_2 be two eigenvectors with distinct eigenvalues λ_1 and λ_2. Prove that $\langle v_1, v_2 \rangle = v_1^* v_2 = 0$

$A v_1 = \lambda_1 v_1$

$A v_2 = \lambda_2 v_2$

$\langle A v_1, A v_2 \rangle = \langle v_1, v_2 \rangle$

$\quad ||$

$\bar{\lambda}_1 \lambda_2 \langle v_1, v_2 \rangle$.

$|\lambda_1| = |\lambda_2| = 1$ since $A \in U(n)$

$\lambda_1 \neq \lambda_2 \Rightarrow \bar{\lambda}_1 \lambda_2 \neq 1$.

So $\langle v_1, v_2 \rangle = 0$

4. (10 points) Prove that two elements A, B in unitary group $U(2)$ are in the same conjugacy class if and only if $trace(A) = trace(B)$ and $det(A) = det(B)$.

"only if" $tr(PAP^{-1}) = tr A$

$\det(PAP^{-1}) = \det A.$

"if". A, B have the same trace and determinant $\Rightarrow$

A, B have the same eigenvalues

eigenvector λ_1, λ_2

Choose $v_1 \neq 0$, s.t. $Av_1 = \lambda_1 v_1$.

$v_1' = \frac{1}{\sqrt{\langle v_1, v_1 \rangle}} v_1$, then $\langle v_1', v_1' \rangle = 1$.

consider $(\mathbb{C} v_1)^\perp$, $\dim = 1$

$\Rightarrow (\mathbb{C} v_1)^\perp = span(v_2)$.

Then $\langle Av_2, Av_1 \rangle = \langle v_2, v_1 \rangle = 0$.

$\qquad \overset{\Vert}{\lambda_1 \langle Av_2, v_1 \rangle} \Rightarrow \langle Av_2, v_1 \rangle = 0.$

$Av_2 \in (\mathbb{C} v_1)^\perp = span(v_2)$, so $Av_2 = \mu v_2$.

$v_2' = \frac{1}{\sqrt{\langle v_2, v_2 \rangle}} v_2$, then $\langle v_2', v_2' \rangle = 1$, so $P = \begin{pmatrix} | & | \\ v_1' & v_2' \\ | & | \end{pmatrix} \in U(2)$, and $P^{-1}AP = \begin{bmatrix} \lambda_1 & \\ & \mu \end{bmatrix}$

5. (10 points) Construct a one dimensional group representation $R: C_n \to GL(1)$ of cyclic group C_n of order n such that $ker(R) = e$.

$\exists Q \in U(2)$,

$P^{-1}AP = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}$

$Q^{-1}BQ = \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix}$, so A, B are in the same conjugacy class

$C_n = \mathbb{Z}/n\mathbb{Z} = \{ \bar{m} = m + n\mathbb{Z} \mid m \in \mathbb{Z} \}$.

$R: C_n \to GL(1) = \mathbb{C}^*$

$\bar{m} \mapsto e^{i\frac{2\pi m}{n}}$

R is well-defind because

if $m_1 \equiv m_2 \bmod n$, then

$e^{i\frac{2\pi m_1}{n}} = e^{i\frac{2\pi m_2}{n}}$

R is group homomorphism because

$R(m_1 + m_2) = e^{i\frac{2\pi}{n}(m_1+m_2)} = R(m_1) \cdot R(m_2)$

6. (20 points) Let V be the vector space of traceless 2×2 real matrices $\{A \in M_{2\times 2}(\mathbb{R}) | trace(A) = 0\}$.
   (a) Prove that $\langle A, B\rangle = trace(A^T B)$ defines a positive definite symmetric bilinear form on V.
   (b) Prove that $P \cdot A = PAP^T$ defines a linear operation of $SO(2)$ on V.
   (c) Use the previous two parts to define a group homomorphism from $SO(2)$ to $SO(3)$.
   (d) Find the kernel of this homomorphism.

a) $V = \{ A \in M_{2\times 2}(\mathbb{R}) \mid \text{trace } A = 0 \}$

has basis $v_1 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$v_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$,

$\langle\ ,\ \rangle$ is bilinear

$\langle A, B_1 + B_2 \rangle = \text{trace}(A^T (B_1 + B_2))$

$= \text{trace} \langle A^T B_1 \rangle + \text{trace}(A^T B_2)$

$= \langle A, B_1 \rangle + \langle A, B_2 \rangle$

$\langle A, cB \rangle = c \langle A, B \rangle$, symmetric. $\langle A, B \rangle = \text{tr}(A^T B)$
$= \text{tr}(B^T A) = \langle B, A \rangle$

$\langle V_1 . V_2 \rangle = 0$, $\langle V_1, V_3 \rangle = 0$, $\langle V_2, V_3 \rangle = 0$

$\langle V_1, V_1 \rangle = 2 > 0$ $\langle V_2 . V_2 \rangle = 1$, $\langle V_3, V_3 \rangle = 1$.
positive definite. > 0 > 0

Draft 1:

If you use this page and want it looked at, then you must indicate so on the page with the original problem on it. Make sure you label your work with the corresponding problem number.

b) $(PQ)\cdot A = (PQ)A(PQ)^T$

$= P(QAQ^T)P^T$

$= P\cdot(Q\cdot A)$

group operation.

$I\cdot A = A$.

$P\cdot(A+B) = P\cdot A + P\cdot B$

$P\cdot(cA) = c(P\cdot A)$

linear.

c) $\langle P\cdot A, P\cdot B\rangle$

$= tr((PAP^T)^T(PBP^T))$

$= tr(PA^TP^TPBP^T)$

$= tr(PA^TBP^T) = tr(P^TPA^TB)$

$= tr(A^T B) = \langle A, B \rangle$

Draft 2:

If you use this page and want it looked at, then you must indicate so on the page with the original problem on it. Make sure you label your work with the corresponding problem number.

$\langle , \rangle$ is preserved by $SO(2)$ operation

So we obtain: $\rho: SO(2) \to O(3)$.

Since $SO(2)$ is path-connected.

$\det \rho(SO(2)) = 1$, $Im\,\rho \subset SO(3)$.

d) $P \in \ker \rho$ iff $PAP^{-1} = A$, $\forall$

$A \in V$.

$$P \cdot \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \cdot P$$

$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\Downarrow$ $b = c = 0$.

$\Rightarrow P = \pm \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$, check $\pm I \in \ker \rho$

so here $\rho = \zeta \pm I\eta$.