Math 371 Homework#2

Due on 2/6 at the beginning of Lecture

1. Prove that $\mathrm{GL}(n, \mathbb{C})$ is isomorphic to a subgroup of $\mathrm{GL}(2n, \mathbb{R})$.

2. Let $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/2 \\ 0 & \sqrt{3}/2 & -1/2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ be elements in $\mathrm{SO}(3)$. Are they in the same conjugacy class? If they are, find $P \in \mathrm{SO}(3)$ such that $A = PBP^{-1}$.

3. Prove that $\mathrm{SO}(n)$ is isomorphic to a subgroup H of $\mathrm{SO}(n+1)$ and the quotient set $\mathrm{SO}(n+1)/H$ (i.e. the set of right cosets) has a bijection with n-dimensional sphere S^n. Use this fact and induction to show that $\mathrm{SO}(n)$ has dimension $n(n-1)/2$. (Hint: use the natural action of $\mathrm{SO}(n+1)$ on S^n and proposition 6.8.4 in Artin.)

4. Artin, Chapter 9, problem 1.2. Here Lorenz group $\mathrm{O}_{3,1}$ is the group of linear transformations on $\mathbb{R}^4$ preserving the symmetric bilinear form with matrix $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, i.e. $\mathrm{O}_{3,1} = \{B \in \mathrm{GL}(4, \mathbb{R}) | B^T A B = A\}$.

5. Let G be the set of maps $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ of the form $f(x) = Ax + y$ with $A \in \mathrm{O}(3)$ and $y \in \mathbb{R}^3$. The multiplication on G is defined by composition of maps.
   (a) Prove that G is a group.
   (b) Prove that $\mathrm{O}(3)$ is a subgroup of G and it is not normal.
   (c) Prove that $(\mathbb{R}^3, +)$ is isomorphic to a normal subgroup of G.

6. Prove that the cyclic group C_n of n-elements is isomorphic to a subgroup H_n of $SO(2)$. Is this H_n unique?

7. Artin, Chapter 9, problem 5.6, the first part finding conjugacy classes.

8. Artin, Chapter 9, problem M.4 a), b). You can try c) but it is not required.