Math 371 Homework#5

Due on 3/6 at the beginning of Lecture

1. Prove the following theorem we stated in the class: the kernel of representation $\rho\colon G \to \mathrm{GL}(V)$ is the same as the kernel of the corresponding character χ, i.e. $\ker\rho = \ker\chi = \{g \in G | \chi(g) = \chi(e)\}$. (Hint: use the following triangle inequality about complex numbers $|a+b| \leq |a| + |b|$ for all $a, b \in \mathbb{C}$. Assuming $b \neq 0$, the equality holds if and only if $a = \lambda b$ with real number $\lambda \geq 0$. Draw a picture of a, b and $a+b$ on complex plane to understand this inequality.)

2. Is a character always a group homomorphism? If not, find one counter example and state in which cases it is a group homomorphism.

3. Let H be a normal subgroup of G. If $\rho\colon G/H \to \mathrm{GL}(V)$ is a representation of G/H. Prove that $\tilde{\rho}\colon G \to \mathrm{GL}(V)$ defined by $\tilde{\rho}(g) = \rho(gH)$ is a representation of G.

4. Let $K_4 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ be the product of two cyclic groups of order 2. Use question 3 to find all the irreducible characters of K_4.

5. Let $\rho\colon S_5 \to \mathrm{GL}(5)$ be the permutation representation of S_5 defined by $\sigma e_i = e_{\sigma(i)}$ where $\{e_1 \cdots e_5\}$ is the standard basis of $\mathbb{C}^5$. Compute the character χ corresponding to ρ and write χ as summation of irreducible characters. Can you guess how many irreducible representations appearing in the direct sum decomposition of permutation representation of S_n?

6. In this question, you will find the character table of A_4 the group of even permutations of S_4. Alternating group A_4 is a subgroup of S_4 and has 4 conjugacy classes

   $$\{(1)\}, \{(12)(34), (23)(14), (13)(24)\}, \{(123), (142), (134), (243)\}, \{(132), (124), (143), (234)\}.$$

   (a) Prove that $K = \{(1), (12)(34), (23)(14), (13)(24)\}$ is a normal subgroup of S_4.

   (b) Prove that A_4/K is cyclic group of order 3. (You can use the fact that any group of prime order is a cyclic group, think about why this is true.)

   (c) Use question 3 to find all the irreducible characters of A_4.

   (d) (Optional) Compare this with character table of S_4 and describe all the irreducible representations of A_4.

7. Artin chapter 10, 4.10 a)

8. Let G be a finite group and $g \in G$. Prove that g and g^{-1} are in the same conjugacy class if and only if $\chi(g)$ is in $\mathbb{R}$ for all characters χ.