Bilinear form:

Defn (Bilinear form): V n-dim'l vector space over $\mathbb{R}$, $\langle\, , \rangle$ is a bilinear form if

$$\langle\, , \rangle : V \times V \to \mathbb{R}.$$
$$(v, w) \mapsto \langle v, w \rangle.$$

$a \in \mathbb{R}$, $v, w \in V$. $v_1, v_2 \in V$.

$$\langle av, w \rangle = a \langle v, w \rangle$$

$$\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle$$

$$\langle w, av \rangle = a \langle w, v \rangle$$

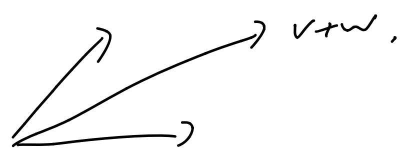
$$\langle w, v_1 + v_2 \rangle = \langle w, v_1 \rangle + \langle w, v_2 \rangle.$$

Defn (Symmetric bilinear form)

$$\langle v, w \rangle = \langle w, v \rangle$$

From now on: $\langle\, , \rangle$ is symmetric.

Prop: $\langle\, , \rangle$ is determined by $\langle v, v \rangle$



$$\langle v, w \rangle = \frac{1}{2}\left(\langle v+w, v+w \rangle - \langle v, v \rangle - \langle w, w \rangle \right)$$

Ex 1: Euclidean space $\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\}$.

$v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $w = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$\langle v, w \rangle = \sum x_i y_i$.

Angle between nonzero v, w,

$$\cos\theta = \frac{\langle v, w \rangle}{\sqrt{\langle v, v \rangle \langle w, w \rangle}}$$

Ex 2: Minkowski space.

$$\langle v, w \rangle = x_1 y_1 + x_2 y_2 \cdots x_{n-1} y_{n-1} - x_n y_n.$$

Thm: There exists basis of V, $v_1 \dots v_n$, such that $v = \sum x_i v_i$.

$$\langle v, v \rangle = x_1^2 + \cdots x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$

$p \cdot q$ are determined by $\langle , \rangle$.

Pf: ① $\langle v, v\rangle = 0$ for all v,

$\Rightarrow \langle v, w\rangle = 0$

② $\exists v$, $\langle v, v\rangle > 0$

③ $\exists v$, $\langle v, v\rangle < 0$.

②: Idea Find basis $v_1 \cdots v_n$

s.t. $\langle v_i, v_j\rangle = 0$ $i \neq j$

$$\langle v_i, v_i\rangle = \begin{cases} 1 & i = 1, \cdots p \\ -1 & i = p+1 \cdots p+q \\ 0 & i > p+q. \end{cases}$$

$$v_1 = \frac{1}{\sqrt{\langle v, v\rangle}} v.$$

$$W = \{ w \in V \mid \langle w, v_1\rangle = 0 \}.$$

Claim: $W \oplus \mathbb{R} v_1 = V$.

ⓐ $W \cap \mathbb{R} v_1 = \{0\}$. If $a v_1 \in W$.

$\langle a v_1, v_1\rangle = a = 0$.

(b) $\forall v \in V$, try $v = w + a v_1$, $w \in W$

$\langle v, v_1 \rangle = a$,

so fix $a = \langle v, v_1 \rangle$

$w = v - a v_1$.

Check $\langle w, v \rangle = 0$. $\Rightarrow w \in W$

Induction on $\dim V$.

Matrix form of $\langle , \rangle$ under basis $v_1 \cdots v_n$. G-S matrix of $\langle , \rangle$

$$A = \begin{pmatrix} & & \\ & \langle v_i, v_j \rangle & \\ & & \end{pmatrix} \leftarrow i\text{th row}$$

$\uparrow$ jth column.

$A = A^T$

$v = \sum x_i v_i$ $\quad w = \sum y_j v_j$.

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$.

$\langle v, w \rangle = x^T A y$.

Prop: If $(w_1 \cdots w_n) = (v_1 \cdots v_n) \cdot P$

is another basis, then

G-S matrix under basis $(w_1 \cdots w_n)$

is $B = P^T A P$.

Thm: For any symmetric real matrix A,

$\exists P$ invertible, s.t.

$$P^T A P = \begin{pmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & -1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & -1 & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{pmatrix}$$

Positive definite $\langle , \rangle$.

Defn: $\langle v, v \rangle > 0$ for all $v \neq 0$.

($P^T A P = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$.)

Prop: $\langle\ ,\ \rangle$ positive definite, W any subspace of V, and $W^\perp = \{v \in V \mid \langle v, w\rangle = 0\ \forall w \in W\}$

then $W \oplus W^\perp = V$.

Pf: ① $W \cap W^\perp = \{0\}$.

$\forall v \in W \cap W^\perp$, $\langle v, v\rangle = 0$.

② choose othornomal basis of W, $v_1 \dots v_m$. $\forall v \in V$.

Assume $v = \sum_{i=1}^{m} a_i v_i + w_2$,

and $w_2 \in W^\perp$,

then $\langle v, v_i\rangle = a_i$.

So fix $a_i = \langle v, v_i\rangle$.

$$w_2 = v - \sum_{i=1}^{m} a_i v_i$$

then check $\langle w_2, v_i\rangle = 0$. $\Rightarrow w_2 \in W^\perp$

Symmetry of $\mathbb{R}^2$. $\langle\ ,\ \rangle$ standard Euclidean positive definite symmetric bilinear form.

$GL(2) = \{$ linear isomorphism of $\mathbb{R}^2 \}$
(bijection)

$= \{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid A \text{ invertible} \}$.

matrix multiplication as group operation.

$O(2) = \{ A \in GL(2) \mid \langle Av, Aw \rangle = \langle v, w \rangle \}$.

In terms of matrix,

$\langle Av, Aw \rangle = v^T \cdot A^T A w$

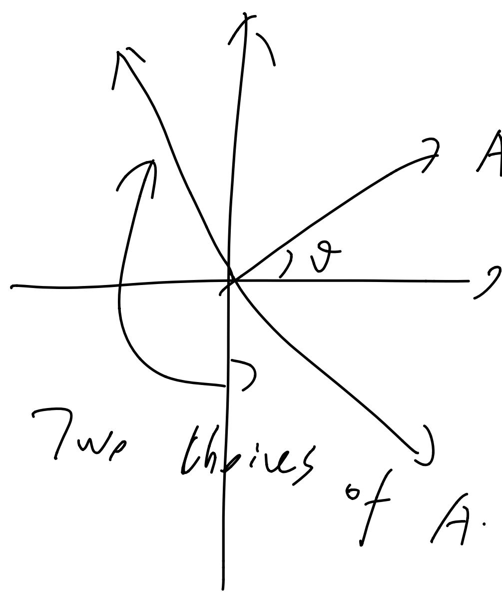
$= v^T w$.

so $A^T A = I$

Equations: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow$$

$$\begin{cases} a^2 + c^2 = 1 \\ ab + cd = 0 \\ b^2 + d^2 = 1 \end{cases}$$



$A \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

Two choices of $A \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$

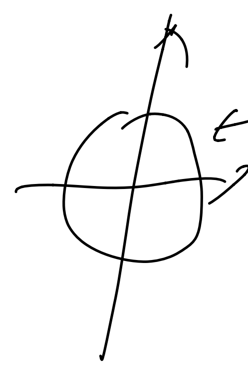
$$O(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle| \ \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}.$$

$$\sqcup \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \middle| \ \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}$$

$SO(2)$ / all the rotations.

all the reflections.

$SO(2)$ commutative.



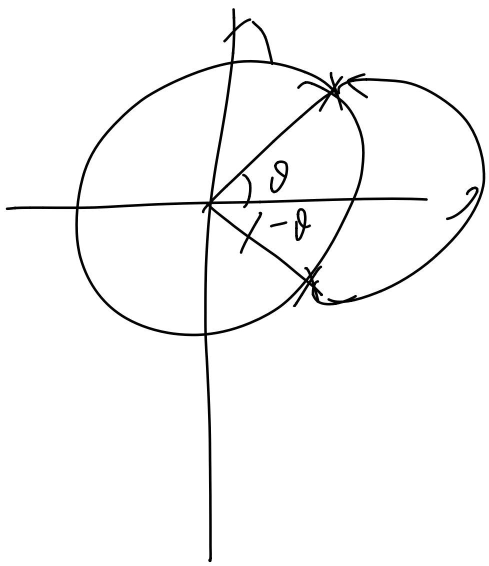
← circle.

$O(2)$ two copies of $SO(2)$

conjugacy classes.

$$\begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(\theta) & \cos(-\theta) \end{bmatrix}.$$



conjugate.

all the reflections are conjugate.