Today's topic.

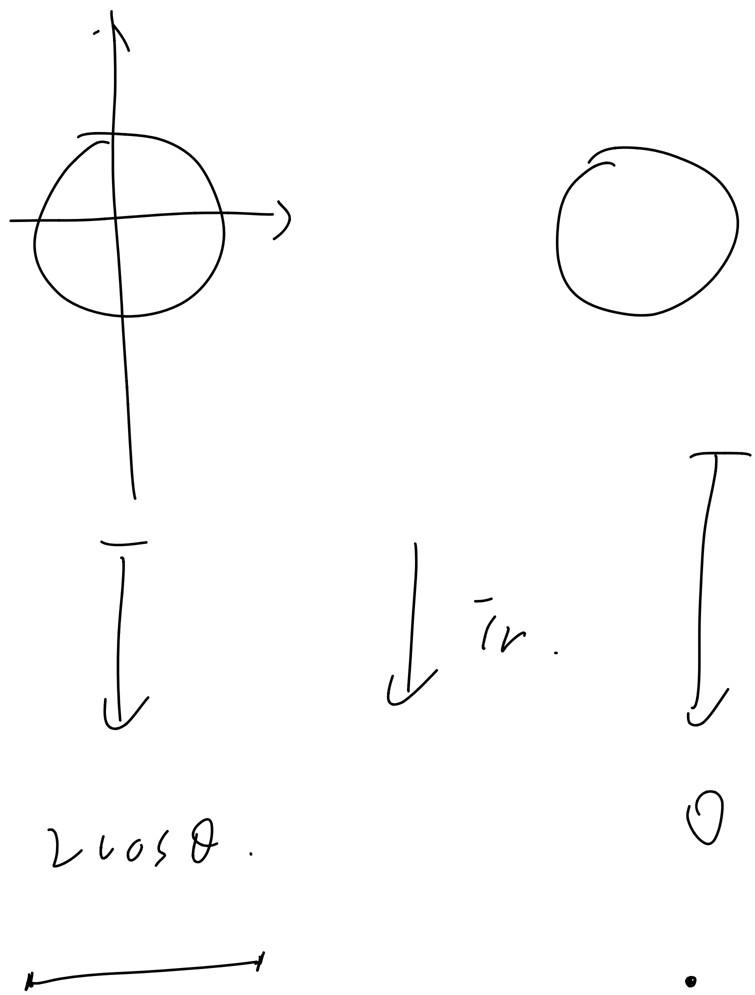
Finish conjugacy classes in $O(2)$

$$A = X_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

$$B = y_{\frac{\theta}{2}} = \begin{bmatrix} \cos\theta & +\sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}.$$

$$\begin{cases} X_{\theta_1} \cdot X_{\theta_2} = X_{\theta_1+\theta_2} \\ y_{\frac{\theta}{2}} = y_0 \cdot X_\theta \\ y_0 \cdot X_\theta \cdot y_0 = X_{-\theta}. \end{cases}$$

$$y_{\frac{\theta}{2}} = y_0 \cdot X_\theta = X_{-\frac{\theta}{2}} \cdot y_0 \cdot X_{\frac{\theta}{2}}.$$



Bilinear forms on $\mathbb{C}^n$.

$$\begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}$$

because we have $\sqrt{}$ of any number.

Hermitian forms $V = V_{\mathbb{C}}$.

$V \times V \to \mathbb{C}$.

$\langle v, w \rangle$

① conjugate linear on v

② linear on w.

③ $\langle v, w \rangle = \overline{\langle w, v \rangle}$.

Prop: $\langle v, v \rangle = \overline{\langle v, v \rangle} \Rightarrow \langle v, v \rangle \in \mathbb{R}$.

Matrix form: $v_1 \dots v_n$ basis.

$$A = \left(\langle v_i, v_j \rangle \right)_{n \times n}$$

$$A = A^* \qquad A^* = \overline{(A^T)}$$

Prop: (change of basis)

$$(w_1, \cdots w_n) = (v_1, \cdots v_n) \cdot P.$$

then $\big((w_i, w_j) \big) = P^* \cdot \big((v_i, v_j) \big) P$

Thm: $\exists$ basis $v_1, \cdots v_n$.

s.t. $(v_i, v_j) = 0$ if $i \neq j$,

and $(v_i, v_i) = \begin{cases} 1 \\ -1 \\ 0 \end{cases}$

Hermitian form is positive definite if and only if $\langle v, v \rangle > 0$, for all $v \neq 0$.

Standard form on $\mathbb{C}^n$.

$\langle v, w \rangle = (\overline{v^T}) w = v^* w$.

$GL(n, \mathbb{C}) = \{ A$ invertible $n \times n$ cplx matrix$\}$.

$U(n) = \{ A \in GL(n, \mathbb{C}) \mid \langle Av, Aw \rangle = \langle v, w \rangle \}$

so $(Av)^* Aw = v^* w \Rightarrow$

$A^* A = I_n$.

$GL(3, \mathbb{R}) = \{ A$ 3×3 invertible real matrix$\}$

$O(3) = \{ A \in GL(3, \mathbb{R}) \mid \underline{\langle Av, Aw \rangle = \langle v, w \rangle} \}$.

$A^T A = I_3$.

$SO(3) = \{ A \in O(3) \mid \det A = 1 \}$.

$\det : GL(3, \mathbb{R}) \to \mathbb{R}^\times$ is a group homomorphism.

$SO(3) = \ker \left(\det\big|_{O(3)} \right)$

$SO(3)$ is a normal subgroup of $O(3)$

since $\det(A^T A) = (\det A)^2$,

so $\operatorname{Im}\left(\det\big|_{O(3)}\right) = \{\pm 1\}$.

$SO(3)$ is an index 2 subgroup of $O(3)$ $\quad \det \begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix} = -1$

Study of $SO(3)$

$A \in SO(3)$, PAP^{-1} has the same eigenvalues as A.

If $\det(\lambda I - A) = 0$ has three cplx roots

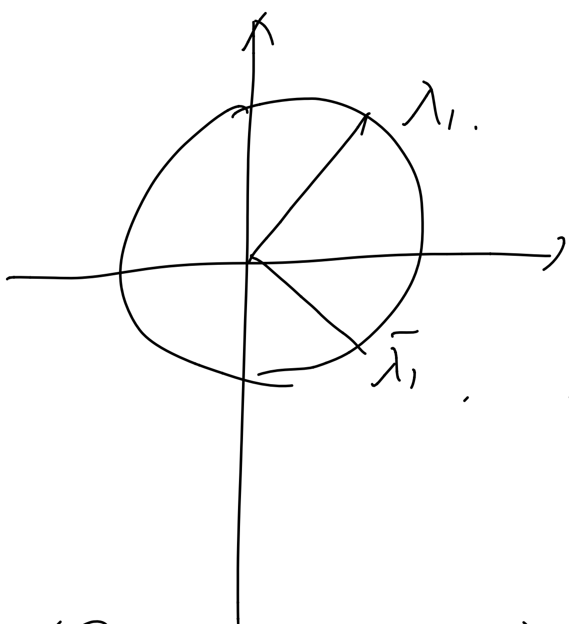
$\lambda_1, \lambda_2, \lambda_3, \in \mathbb{C}$

λ_1 has eigenvector $v \in \mathbb{C}^n$.

then $\langle Av, Av \rangle = \langle v, v \rangle$

$$|\lambda_1|^2 \langle v, v \rangle = \langle v, v \rangle$$

$\Rightarrow |\lambda_1|^2 = 1$. So are λ_2, λ_3.



Since $|\lambda I - A|$ is a real polynomial, $\bar{\lambda}_1$ is also a root.

① $\lambda_1 \neq \pm 1$, $\lambda_1 \neq \bar{\lambda}_1$,

so we can assume $\lambda_2 = \bar{\lambda}_1$.

$\lambda_1 \lambda_2 \lambda_3 = 1 \Rightarrow \lambda_3 = 1$

② $\lambda_1 = 1$

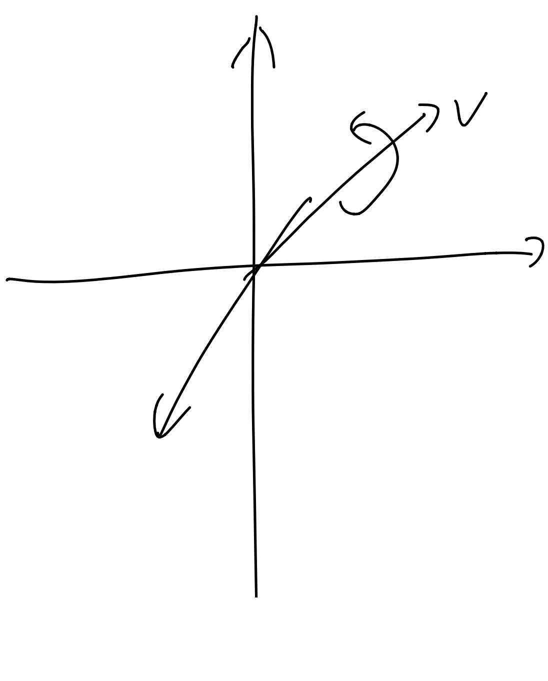
③ $\lambda_1 = -1$, then. $\lambda_2 = \pm 1$ or $\lambda_2 \neq \pm 1$,

all implies there is one $\lambda_i = 1$.

so $|\lambda I - A|$ has one root 1.

and a real eigen vector $v \in \mathbb{R}^n$.

$Av = v$.



we will see next time A is a rotation along vector v.