General notions of Lie groups.

1. General linear groups

$GL(n, \mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) \mid A \text{ invertible}\}$.

$GL(n, \mathbb{C}) = \{A \in M_{n\times n}(\mathbb{C}) \mid A \text{ invertible}\}$.

$GL(n, \mathbb{R}) \subset GL(n, \mathbb{C})$ is a subgroup.

2. Special linear group.

$SL(n, \mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) \mid \det A = 1\}$.

3. Orthogonal group.

$O(n) = \{A \in M_{n\times n}(\mathbb{R}) \mid A^T A = 1\}$.

4. Special orthogonal group

$SO(n) = \{A \in M_{n\times n}(\mathbb{R}) \mid A^T A = 1, \det A = 1\}$

5. Unitary group

$U(n) = \{A \in M_{n\times n}(\mathbb{C}) \mid A^* A = 1\}$

$A^* = (\bar{A})^T$

6. Special unitary group

$SU(n) = \{A \in U(n) \mid \det A = 1\}$.

HW: $GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$ is a subgroup of $GL(2n, \mathbb{R})$

Hint: $\mathbb{C}^n \cong \mathbb{R}^{2n}$ as real vector space.

Prop: All eigenvalues λ of $A \in U(n)$ has $|\lambda| = 1$.

Pf: Let $v \in \mathbb{C}^n$ be a non-zero eigenvector. s.t. $Av = \lambda v$.

$\langle Av, Av \rangle = \langle v, v \rangle$

$\langle Av, Av \rangle = \langle \lambda v, \lambda v \rangle = |\lambda|^2 \langle v, v \rangle$.

so $|\lambda|^2 = 1$.

Corollary. Let $A \in SO(n)$ and λ is an eigenvalue of A,

then $|\lambda| = 1$ and $\bar{\lambda}$ is also an eigenvalue of A.

pf: $|\lambda I - A|$ is a real polynomial.

Conjugacy class in $SO(3)$

From Lecture 3, we know $\lambda = 1$ is an eigenvalue of $A \in SO(3)$.

Then there exists an eigenvector $v \in \mathbb{R}^3$,

$Av = v$.

Let $W = (\mathbb{R}v)^\perp$, then $\mathbb{R}v \oplus W = \mathbb{R}^3$

and

Prop: A preserves W.

Pf: Let $w \in W$.

$\langle Aw, Av \rangle = \langle w, v \rangle = 0$

$\langle Aw, v \rangle = 0$

$\Rightarrow Aw \in W$.

So A is an orthogonal operator on W.

So there exists orthonormal basis of W. v_2, v_3.

s.t. $A(v_2, v_3) = (v_2, v_3) \cdot \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta, & \cos\theta \end{bmatrix} = \tilde{A}$

$v_1 = \frac{1}{|v|} v$.

or $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

Since $A(v_1, v_2, v_3)$

$= (v_1, v_2, v_3) \cdot \begin{bmatrix} 1 & \\ & \tilde{A} \end{bmatrix}$

and $\det A = 1$, so $\det \tilde{A} = 1$

$$\tilde{A} = \begin{bmatrix} \cos\theta, & -\sin\theta \\ \sin\theta, & \cos\theta \end{bmatrix}$$

Let $P = (v_1, v_2, v_3)$. then $P^T P = I_n$.

If $\det P = -1$, replace v_1 by $-v_1$

then $\det P = 1$.

So $$P^{-1} A P = \begin{bmatrix} 1 & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{bmatrix}$$

Thm: The conjugacy class of $SO(3)$ is determined by trace function

Pf: Every A can be conjugate to

$$P^{-1} A P = \begin{bmatrix} 1 & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{bmatrix}.$$

If $$A_1 = \begin{bmatrix} 1 & & \\ & \cos\theta_1 & -\sin\theta_1 \\ & \sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & & \\ & \cos\theta_2 & -\sin\theta_2 \\ & \sin\theta_2 & \cos\theta_2 \end{bmatrix}$$

and $\operatorname{tr} A_1 = \operatorname{tr} A_2$.

then $\theta_1 = \theta_2$ or $\theta_1 = -\theta_2$.

If $\theta_1 = -\theta_2$, $\theta_1 \neq \theta_2$.

Choose $\tilde{P} \in O(2)$, $\det \tilde{P} = -1$.

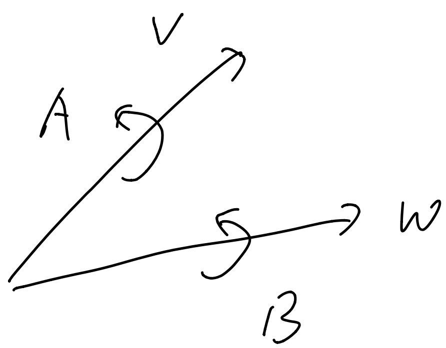
s.t. $\tilde{P} \begin{bmatrix} \cos\theta_1, & -\sin\theta_1 \\ \sin\theta_1, & \cos\theta_1 \end{bmatrix} \tilde{P}^{-1}$

$$= \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2, & \cos\theta_2 \end{bmatrix}.$$

$$P = \begin{bmatrix} -1 & \\ & \tilde{P} \end{bmatrix}.$$

then $P A_1 P^{-1} = A_2$.

Geometrically. If $Av = v$ and
$Bw = w$



A is an rotation of degree θ counterclockwise from positive v direction.

B - - - - - θ - - - - positive w direction.

Let $P \in SO(3)$ such that
$Pv = w$.

then $PAP^{-1} = B$.

More generally:

Thm: Let $A \in SO(n)$, then $\exists P \in SO(n)$

s.t. $$PAP^{-1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & & & \\ \sin\theta_1 & \cos\theta_1 & & & \\ & & \cos\theta_2 & -\sin\theta_2 & \\ & & \sin\theta_2 & \cos\theta_2 & \\ & & & & \ddots \\ & & & & & 1 \\ & & & & & & 1 \end{bmatrix}$$

Pf: Let λ is an eigenvalue of A.

If $\lambda = 1$, then Let $v \neq 0$,

$Av = v$, and choose $W = (\mathbb{R}v)^{\perp}$

induction on W.

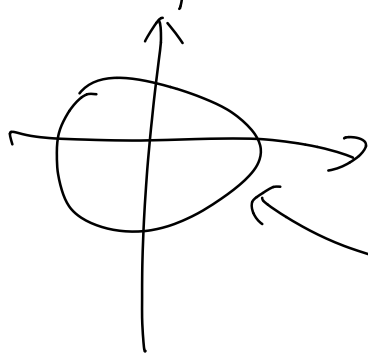
If $\lambda = -1$, then there exists another eigenvalue $\lambda' = -1$. otherwise $\det A \neq 1$. $(\lambda \cdot \bar{\lambda} = 1)$

If $\lambda \neq \{\pm 1\}$. λ is not real.

$A \cdot (v_1 + \sqrt{-1}\, v_2) = \lambda\, (v_1 + \sqrt{-1}\, v_2)$, $v_1, v_2 \in \mathbb{R}^n$.

Take $\operatorname{span}(v_1, v_2)$.

Dimension of $SO(2)$ is 1.



$SO(2) \cong S^1$

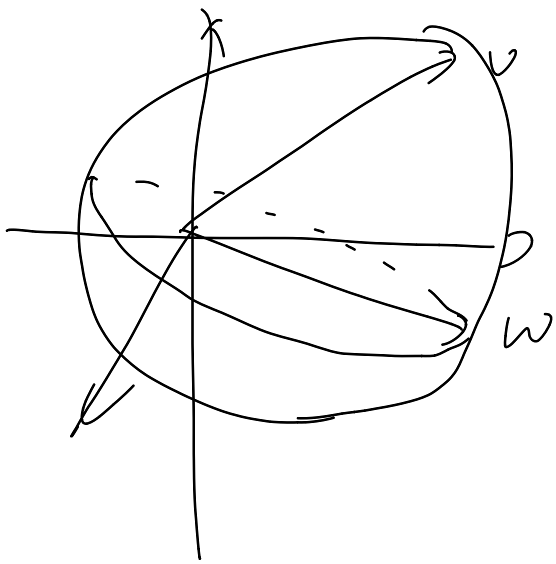
$U(1) = \{\lambda \in \mathbb{C}^* \mid \bar{\lambda}\lambda = 1\} \cong SO(2)$

$$\theta \longmapsto \begin{bmatrix} \cos\theta, & -\sin\theta \\ \sin\theta, & \cos\theta \end{bmatrix}$$

$U(1) \cong SO(2) \cong S^1$

1 dim'l sphere.

$SO(3)$ $A = [v, w, v \times w]$.



① choice of v,
2-dim'l sphere

② choice of w
circle perpendicular to v.
1-dim'l circle.

③ $v \times w$

Prop: $\dim SO(3) = 2 + 1 = 3$.

Why important.

Noether's Thm: Every symmetry gives a conservation Law.

$SO(3) \Rightarrow$ angular momentum. (3 components

time translation $\Rightarrow$ energy

$(\mathbb{R}^3, +)$ space translation $\Rightarrow$ momentum.

(3 components)

$$SU(2) = \left\{ A = \begin{pmatrix} a, & b \\ c, & d \end{pmatrix} \;\middle|\; \begin{array}{l} A^* A = I_2 \\ \det A = 1 \end{array} \right\}.$$

$$A^* = A^{-1}$$

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^*$$

so $a = \bar{d}$, $\quad c = -\bar{b}$.

$$\det \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} = |a|^2 + |b|^2 = 1$$

$a = x + y\sqrt{-1}, \quad b = z + w\sqrt{-1}$

$x, y, z, w \in \mathbb{R}$,

then $x^2 + y^2 + z^2 + w^2 = 1$.

Prop: $SU(2) = \left\{ A = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \middle| \, |a|^2 + |b|^2 = 1 \right\}$.

Pf: $\left. \begin{array}{l} A^* A = I_2 \\ \det A = 1 \end{array} \right\} \Rightarrow A = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$

$|a|^2 + |b|^2 = 1$

$\left. \begin{array}{l} A = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \\ |a|^2 + |b|^2 = 1 \end{array} \right\} \Rightarrow \begin{array}{l} A^* A = I_2 \\ \det A = 1. \end{array}$

Conjugacy classes in $SU(2)$.

Prop: Given $A \in SU(2)$, $\exists P \in SU(2)$ s.t. $P^{-1} A P = \begin{bmatrix} \lambda & \\ & \bar{\lambda} \end{bmatrix}$ with $|\lambda| = 1$.

Pf: $P = [v_1, v_2]$

then $P v_1 = \lambda v_1$, $P v_2 = \bar{\lambda} v_2$

$$v_1 = \begin{pmatrix} a \\ b \end{pmatrix}, \quad v_2 = \begin{pmatrix} -\bar{b} \\ \bar{a} \end{pmatrix}$$

Let λ be an eigenvalue of A with eigenvector v. then $Av = \lambda v$. Let $v_1 = \frac{1}{|v|} v$, then

$$v_1 = \begin{bmatrix} a \\ b \end{bmatrix} \text{ and } |a|^2 + |b|^2 = 1.$$

Let $v_2 = \begin{bmatrix} -\bar{b} \\ \bar{a} \end{bmatrix}$, then

$\langle v_1, v_2 \rangle = 0$, v_2 is the basis of

$(\mathbb{C} v_1)^\perp$.

A preserves $\langle , \rangle$, so $A v_2 \in (\mathbb{C} v_1)^\perp$.

(why? $\langle A v_1, A v_2 \rangle = \langle v_1, v_2 \rangle = 0$

$= \bar{\lambda} \langle v_1, A v_2 \rangle$

$\Rightarrow \langle v_1, A v_2 \rangle = 0$.)

so $A v_2 = \mu v_2$.

Since $A \cdot [v_1, v_2] = [v_1, v_2] \cdot \begin{bmatrix} \lambda & \\ & \mu \end{bmatrix}$

then $\lambda \mu = 1$, $\Rightarrow \mu = \bar{\lambda}$.

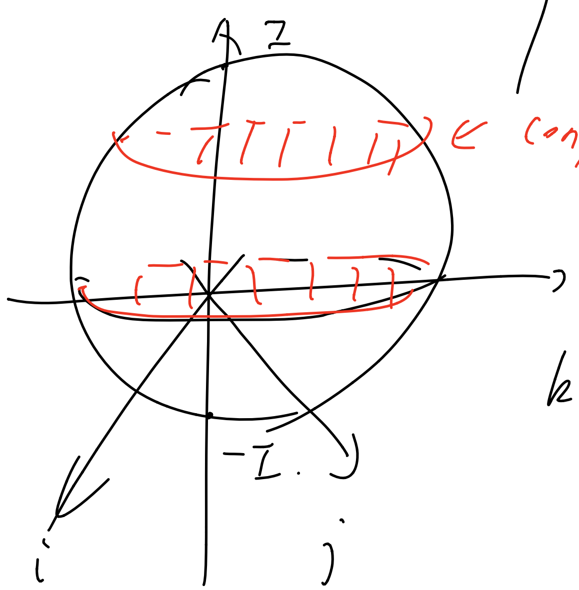
Prop: Conjugacy classes of $SU(2)$ is determined by trace function.

$\dim SU(2) = 3$.

$$I = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \quad i = \begin{bmatrix} \sqrt{-1} & \\ & -\sqrt{-1} \end{bmatrix}, \quad j = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}.$$

$$k = \begin{bmatrix} & \sqrt{-1} \\ \sqrt{-1} & \end{bmatrix}$$ basis of

space of matrices $\left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \right\} \cong \mathbb{R}^4$.



$\leftarrow$ conjugacy class $\cong S^2$.

$SU(2) \cong S^3$.

Each conjugacy class is $\cong$ to S^2 except $\{I\}$, and $\{-I\}$.

Goal to prove:

There exists homomorphism

$f: SU(2) \to SO(3)$ such that

$\ker f = \{\pm I\}$.

One parameter subgroup of $SU(2)$

Prop: There exists subgroup $\left\{\begin{bmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{bmatrix}\right\}$ of $SU(2)$, isomorphic to $U(1) \cong SO(2)$

$$V = \left\{ B = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \,\middle|\, \operatorname{tr} B = 0 \right\}.$$

Prop: $SU(2)$ acts on V via:

$$SU(2) \times V \to V$$
$$(A, B) \mapsto ABA^{-1}.$$

Claim:

$$V = \left\{ B \in M_{2\times 2}(\mathbb{C}) \,\middle|\, B^* + B = 0,\ \operatorname{tr} B = 0 \right\}.$$

$$\begin{bmatrix} a & b \\ -\bar{b}, & \bar{a} \end{bmatrix} + \begin{bmatrix} \bar{a} & \bar{b} \\ -b & a \end{bmatrix} = \begin{bmatrix} a+\bar{a} & \\ & a+\bar{a} \end{bmatrix}$$

$\operatorname{tr} B = a + \bar{a} = 0$

$$\Rightarrow \begin{cases} B^* + B = 0 \\ \operatorname{tr} B = 0 \end{cases}$$

$$\begin{cases} B^* + B = 0 \\ \operatorname{tr} B = 0 \end{cases} \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix} = 0$$

$a + d = 0$

$$\begin{cases} a + \bar{a} = b + \bar{c} = c + \bar{b} = d + \bar{d} = 0 \\ a + d = 0 \end{cases}$$

$$\Rightarrow \quad a = \bar{d}, \quad b = -\bar{c}, \quad a + \bar{a} = 0.$$

So $tr(A B A^{-1}) = 0$

$$A B A^{-1} = A B A^*.$$

$$(A B A^*)^* + A B A^*$$

$$= A B^* A + A B A^*$$

$$= A (B^* + B) A^* = 0.$$

Next time: we will show $SU(2)$ preserves an inner product on $V \cong \mathbb{R}^3$.