SAMPLE SOLUTIONS HW 2

HW 2, Problem 1

Prove that $\mathrm{GL}_n(\mathbb{C})$ *is isomorphic to a subgroup of* $\mathrm{GL}_{2n}(\mathbb{R})$.

Proof. Note that $\mathbb{C}^n$ is an $\mathbb{R}$-vector space in the obvious sense (i.e., given $\alpha \in \mathbb{R}$, $(z_1, \ldots, z_n) \in \mathbb{C}^n$, $\alpha(z_1, \ldots, z_n) = (\alpha z_1, \ldots, \alpha z_n)$). Let $\varphi : \mathbb{C}^n \to \mathbb{R}^{2n}$ be given by sending $(a_1 + ib_1, \ldots, a_n + ib_n) \mapsto (a_1, b_2, \ldots, a_n, b_n)$. One checks immediately that this map is $\mathbb{R}$-linear and admits the $\mathbb{R}$-linear inverse $\mathbb{R}^{2n} \to \mathbb{C}^n : (a_1, b_1, a_2, b_2, \ldots, a_n, b_n) \mapsto (a_1 + ib_1, \ldots, a_n + ib_n)$. Hence we have a map $\Phi : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_{2n}(\mathbb{R})$ given by $\Phi(\psi) = \varphi \circ \psi \circ \varphi^{-1}$. This map is well-defined (i.e., $\varphi \circ \psi \circ \varphi^{-1} \in \mathrm{GL}_{2n}(\mathbb{R})$) since φ and ψ are both isomorphisms of $\mathbb{R}$-vector spaces.

It remains to show that Φ is an injective group homomorphism. For $\psi_1, \psi_2 \in \mathrm{GL}_n(\mathbb{C})$, we have

$$\Phi(\psi_1 \circ \psi_2) = \varphi \circ (\psi_1 \circ \psi_2) \circ \varphi^{-1} = (\varphi \circ \psi_1 \circ \varphi^{-1}) \circ (\varphi \circ \psi_2 \circ \varphi^{-1}) = \Phi(\psi_1) \circ \Phi(\psi_2),$$

whence Φ is a group homomorphism. If $\Phi(\psi) = id_{\mathbb{R}^{2n}}$, then $\varphi \circ \psi \circ \varphi^{-1} = id_{\mathbb{R}^{2n}}$, whence $\psi = id_{\mathbb{C}^n}$ (by multiplying on the right by φ and on the left by φ^{-1} of both sides of the equality). □

HW 2, Problem 6

Prove that the cyclic group C_n *of* n *elements is isomorphic to a subgroup* H_n *of* $SO(2)$*. Is this* H_n *unique?*

Proof. Identify $SO(2)$ with $S^1 = \{e^{i\theta} \in \mathbb{C}^\times : \theta \in \mathbb{R}\}$ (with group structure induced by that of $\mathbb{C}^\times$, i.e., multiplication of complex numbers) as usual. A copy of C_n inside S^1 is given by the subgroup generated by $e^{2\pi i/n}$ (precisely, what this means is that there is an *injective group homomorphism* $C_n \hookrightarrow S^1$ with image $\langle e^{2\pi i/n} \rangle$, the subgroup generated by $e^{2\pi i/n}$).

Now, if $G \subseteq S^1$ is any other subgroup of order n, then for any $g \in G$, we have $g^n = 1$. If $g = e^{i\theta}$, then $e^{i\theta n} = 1$, so that $n\theta = 2\pi k$ for some integer k. Hence $\theta = 2\pi k/n$, so that $g \in \langle e^{2\pi i/n} \rangle$. Hence $G \subseteq \langle e^{2\pi i/n} \rangle$, and since their cardinalities are equal, these two groups are equal. In other words, we have shown that $\langle e^{2\pi i/n} \rangle$ is the *unique subgroup of order* n in S^1 (hence in $SO(2)$). □

Date: February 17, 2020.