SAMPLE SOLUTIONS HW 3

HW 3, Artin 9.4.3

Extend the orthogonal representation $\varphi : SU(2) \to SO(3)$ to a homomorphism $\Phi : U(2) \to SO(3)$, and describe the kernel of Φ.

Proof. Let V be the set of 2×2 trace zero, skew-Hermitian matrices; i.e., $V = \{A \in SU(2) : \text{trace}(A) = 0, A = -A^*\}$. V is a 3-dimensional $\mathbb{R}$-vector space with basis

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

If $P \in U(2)$ and $A \in V$, then PAP^{-1} also has trace zero, and $(PAP^{-1})^* = PA^*P^{-1} = P(-A)P^{-1} = -PAP^{-1}$ (where we use the fact that $P^{-1} = P^*$ for $P \in U(2)$), whence $PAP^{-1} \in V$. This defines an action of $U(2)$ on V, which we denote by $*$, since $I * A = IAI^{-1} = A$, and $(PQ) * A = (PQ)A(PQ)^{-1} = (PQ)A(Q^{-1}P^{-1}) = P(QAQ^{-1})P^{-1} = P * (Q * A)$.

Hence, we get a map $\Phi : U(2) \to GL(V) = GL_3(\mathbb{R})$ (we identify $GL(V) = GL_3(\mathbb{R})$ using the basis i, j, k). It remains to check that the image of Φ lies in $SO(3)$.

Via the isomorphism of $\mathbb{R}$-vector spaces $\mathbb{R}^3 \cong V$ via the basis i, j, k, the usual inner product on $\mathbb{R}^3$ induces a natural symmetric positive definite bilinear form on V given by $\langle A, B\rangle = a_1b_1 + a_2b_2 + a_3b_3$, where $A = a_1i + a_2j + a_3k$, and similarly for B. One verifies by a direct calculation that $\langle A, B\rangle = -\frac{1}{2}\text{trace}(AB)$. Now take any $P \in U(2)$. Note that $SO(3) = \{M \in GL(V) : \langle MA, MB\rangle = \langle A, B\rangle, \forall A, B \in V, \text{and } \det(M) = 1\}$. Hence to show $\Phi(P) \in SO(3)$ we must show that for all $A, B \in V$, we have:
(i) $\langle \Phi(P)A, \Phi(P)B\rangle = \langle A, B\rangle$, and
(ii) $\det(\Phi(P)) = 1$.

(i) follows from the following calculation:

$$\begin{aligned} \langle \Phi(P)A, \Phi(P)B\rangle &= -\frac{1}{2}\text{trace}((\Phi(P)A)(\Phi(P)B)) \\ &= -\frac{1}{2}\text{trace}(PAP^*PBP^*) \\ &= -\frac{1}{2}\text{trace}(PABP^*) \\ &= -\frac{1}{2}\text{trace}(AB) \\ &= \langle A, B\rangle, \end{aligned}$$

where we use the fact that $P^{-1} = P^*$ since $P \in U(2)$. This shows that $\Phi(P) \in O(3)$, so that $\det(\Phi(P)) = \pm 1$. One can show that $U(2)$ is homeomorphic to $S^3 \times S^1$ (this is also exercise 9.3.2 in Artin), and in particular $U(2)$ is connected (since both S^3 and S^1 are). We have a continuous group homomorphism $U(2) \xrightarrow{\Phi} O(3) \xrightarrow{\det} \{\pm 1\}$ (where the set $\{\pm 1\}$ has the discrete topology. The continuous image of a connected set is connected, whence the image of $U(2)$ is either $\{1\}$ or $\{-1\}$. Since $id \in U(2)$ mapsto 1, we conclude that the image is 1, and hence $\det(\Phi(P)) = 1$ for all $P \in U(2)$, and Φ has image in $SO(3)$, as required. Finally, since $\varphi : SU(2) \to SO(3)$ is defined also via a conjugation action on V, it is clear that Φ extends φ. □

Date: February 17, 2020.

HW 3, Problem 6

Let $W = \{A \in M_3(\mathbb{R}) : A = -A^t\}$. Then $P * A = PAP^t$ defines an operation (i.e., a group action) of $SO(3)$ on W. Find a positive definite symmetric bilinear form on W which is invariant under this operation.

Proof. To see that $P * A = PAP^t$ is a group action, we first note that PAP^t is indeed an element of W (i.e., the action is well-defined). For this, note that $(PAP^t)^t = PA^tP^t = P(-A)P^t = -PAP^t$, whence $PAP^t \in W$. Next, we note that $I * A = IAI^t = A$, and that given $P, Q \in SO(3)$, we have

$$(PQ) * A = (PQ)A(PQ)^t = (PQ)A(Q^tP^t) = P(QAQ^t)P^t = P * (Q * A).$$

Hence the above operation is indeed an action of $SO(3)$ on W.

Next, we find the form. Every element of W is of the form

$$M_{a,b,c} := \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

with $a, b, c \in \mathbb{R}$. Using this, one computes that $-\frac{1}{2}\text{trace}(M_{a,b,c}M_{a',b',c'}) = aa' + bb' + cc'$. One checks that this form is a positive definite symmetric bilinear form on W. It is also invariant under the operation since

$$\begin{aligned} -\frac{1}{2}\text{trace}\left((P * M_{a,b,c})(P * M_{a',b',c'})\right) &= -\frac{1}{2}\text{trace}\left(PM_{a,b,c}P^tPM_{a',b',c'}P^t\right) \\ &= -\frac{1}{2}\text{trace}\left(PM_{a,b,c}M_{a',b',c'}P^t\right) \\ &= -\frac{1}{2}\text{trace}\left(M_{a,b,c}M_{a',b',c'}\right), \end{aligned}$$

using the fact that $P^{-1} = P^t$ for $P \in SO(3)$. □