

代数 1 H 班 作业 10

2023 年 8 月 3 日

题 1. *Artin, Chapter 14, 7.3 (任选两个小问做即可)*

题 2. *Artin, Chapter 14, 7.7*

题 3. *Artin, Chapter 14, 7.8*

题 4. *Artin, Chapter 14, 7.9*

题 5. *Let G be a finite abelian group and d be the maximal number of orders of elements in G . Prove that the orders of elements in G divide d . Use this fact to show that every finite subgroup of multiplicative group F^\times in a field F is cyclic.*

题 6. *Artin, Chapter 14, 8.1*

题 7. *Artin, Chapter 14, 8.2*

题 8. *Artin, Chapter 14, 8.4. (注意, 这里表示矩阵 (presentation matrix) 是将生成元的每条关系的系数按行向量排在一起的矩阵, 如果是 n 个生成元, m 条关系, 则该矩阵为 $m \times n$ 矩阵。)*

题 9. *Artin, Chapter 14, 8.6*

题 10. *Let G be the group $\text{GL}(3, \mathbb{F}_2)$.*

1. *How many conjugacy classes does G have?*

2. *Show that G has exactly two conjugacy classes of size 24.*

题 11. *Show that for finitely generated $R = \mathbb{Z}[\sqrt{-5}]$ -modules, torsion-freeness is not equivalent to freeness.*

题 12. Find the number of subgroups of order 27 in $\mathbb{Z}/27 \times (\mathbb{Z}/9)^2 \times (\mathbb{Z}/3)^3$. You only need to write down a formula without evaluating it.

题 13. 1. For a finite \mathbb{Z} -module $M \cong \mathbb{Z}/9 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$, find the number of triples of submodules $M_1 \cong \mathbb{Z}/9, M_2 \cong \mathbb{Z}/3, M_3 \cong \mathbb{Z}/3$, such that $M = M_1 \oplus M_2 \oplus M_3$.

2. (思考题, 比较复杂, 学有余力且觉得这个问题有趣时选做) For a general module M over PID, we know that M is isomorphic to direct sum of cyclic modules $R/(p_i^{n_{ij}})$ with elementary divisors $p_i^{n_{ij}}$. How to classify the tuples of cyclic submodules of M realizing this direct sum? For example, for a finite abelian group, count the number of such decompositions.

题 14. 1. Show that any invertible matrix over Euclidean domain is the product of elementary matrices.

2. (思考题, 比较复杂, 学有余力且觉得这个问题有趣时选做) Show that the first statement does not hold for $R = \mathbb{Z}[\frac{\sqrt{-19}+1}{2}]$. reference: Cohn, P. M. (1966). On the structure of the GL_2 of a ring. Publications Mathématiques de l'IHÉS, 30, 5-53.

题 15.

定义 1. Let R be a PID and $A \in M_{m \times n}(R)$. The determinant of submatrix with i_1, i_2, \dots, i_k th rows and j_1, j_2, \dots, j_k th columns is called a $k \times k$ -minor of A . The greatest common divisor of all $k \times k$ -minors is called a determinant divisors a_k .

1. When R is ED, show that a_k does not change when A is multiplied by invertible matrices on the left or right.

2. (选做, 不用交) Show that this is also true for R being PID.

3. Show that determinant divisors and invariant factors determines each other.

题 16. A module is called semi-simple if it is direct sum of simple modules. Let R be a PID, classify finitely generated semi-simple modules. Show that any submodule of a semi-simple module is still semi-simple.

题 17 (选做题, 比较复杂, 学有余力且觉得这个问题有趣时选做). 试分类 $R = \mathbb{Z}[\sqrt{-5}]$ 上的有限生成模.