代数 1 H 班 作业 11

2023 年 8 月 3 日

题 1. *Let A be a 3×3 matrix with elements in rational numbers and $A^5 = I$. Find all the possible A.*

题 2. *Let $A \in M_n(F)$ be a $n\times n$ matrix over a field F. Let $D \subset M_n(F)$ be the set of matrices commuting with A.*

1. *Under what conditions on its elementary divisors, the set D consists of only polynomials of A. Try to find sufficient and necessary conditions.*
2. *Under what conditions on its elementary divisors, any nonzero matrix $B \in D$ is invertible. Try to find sufficient and necessary conditions.*

题 3. *(丘赛 2011) Let V be a finite-dimensional vector space over $\mathbb{R}$ and $T\colon V \to V$ be a linear transformation such that*

1. *the minimal polynomial of T is irreducible;*
2. *there exists a vector $v \in V$ such that $\{T^i v \mid i \geq 0\}$ spans V.*

Show that V contains no non-trivial proper T-invariant subspace.

题 4. *Let T a linear transformation on vector space V. Under what conditions on its elementary divisors, any T-invariant subspace W has an T-invariant subspace W' such that $W \oplus W' = V$.*

题 5. *(Harvard Qualifying exam 1996) Let $M \in \mathcal{M}_n(\mathbb{C})$ be a complex $n\times n$ matrix such that M is similar to its complex conjugate $\bar{M}$; i.e., there exists $g \in GL_n(\mathbb{C})$ such that $\bar{M} = gMg^{-1}$. Prove that M is similar to a real matrix $M_0 \in \mathcal{M}_n(\mathbb{R})$.*

题 6. *Prove that two matrices A and B over rational numbers are similar in $M_n(\mathbb{Q})$ if and only if they are similar in $M_n(\mathbb{C})$.*

题 7. 1. *Prove that the square root of $A \in M_n(\mathbb{C})$ exists if A is invertible.*

2. *For $n \geq 2$, give examples of $A \in M_n(\mathbb{C})$ such that the square root of A does not exist.*

Counter example in functional analysis. (From Sam Walters' twitter)

Given any bounded connected open subset D of the complex plane $\mathbb{C}$, one has the complex Hilbert space $\mathcal{H}(D)$ of all analytic functions $f : D \to \mathbb{C}$ that are square integrable and with inner product $\|f\|_2^2 = \int_D |f(z)|^2 dxdy < \infty$, $\langle f, g \rangle = \int_D f(z)\overline{g(z)}dxdy$ where $dxdy$ is the usual area (Lebesgue) measure, and $f, g \in \mathcal{H}(D)$. On this Hilbert space the position operator $Q : \mathcal{H}(D) \to \mathcal{H}(D)$ is defined by $(Qf)(z) = zf(z)$ (which simply multiplies $f(z)$ by z). It is easy to show that Q is a continuous linear operator.

定理 1 (Halmos-Lumer-Schaffer, 1953)**.** *The position operator Q has a (continuous) operator square-root if, and only if, $\sqrt{D}$ is a disconnected set.*

Here, the square-root of the set D is $\sqrt{D} := \{z \in \mathbb{C} : z^2 \in D\}$. The theorem implies that if D is chosen so that $\sqrt{D}$ is connected, then Q has no square-root. Example: if D is the open ring $D = \{z \in \mathbb{C} : 1 < |z| < 2\}$, its square-root is $\sqrt{D} = \{w \in \mathbb{C} : 1 < |w| < \sqrt{2}\}$ which is connected. In this case it follows that the operator Q is invertible and has no square-root. In particular, Q cannot be an exponential $\left(Q \neq e^T \text{ for any operator } T\ \right)$.

题 8. *Let A be square matrix over a field. Show that A and A^T are similar over this field.*

题 9. *Let g be a bilinear form on a real vector space V. Prove that if g satisfies $g(x, y) = 0$ if and only if $g(y, x) = 0$, then g is either symmetric or alternating.*

题 10. *Artin chapter 8 1.1. Show that a bilinear form $\langle , \rangle$ on a real vector space V is a sum of a symmetric form and a skew-symmetric form. (skew-symmetric means alternating)*

题 11. *Let F be field that $2 \neq 0$ with automorphism σ of order 2. Denote by $\sigma(z) = \bar{z}$. Field theory tells us that there is an element a in F such that $\sigma(a) = -a$ (You can try to prove this yourself if you are interested). Let g be Hermitian form (skew-Hermitian form) on a F-vector space V, prove that $a \cdot g$ is a skew-Hermitian form (Hermitian form). Here skew-Hermitian form means $B(v, w) = -\overline{B(w, v)}$.*

题 12. *Lang Chapter XV, 1. Here we choose σ to be complex conjugation.*

1. *Let E be a finite dimensional space over the complex numbers, and let*

   $$h : E \times E \to \mathbb{C}$$

   be a hermitian form. Write

   $$h(x, y) = g(x, y) + if(x, y)$$

   where g, f are real valued. Show that g, f are $\mathbb{R}$-bilinear, g is symmetric, f is alternating.

2. *Let E be finite dimensional over $\mathbf{C}$. Let $g : E \times E \to \mathbf{C}$ be $\mathbb{R}$-bilinear. Assume that for all $x \in E$, the map $y \mapsto g(x, y)$ is $\mathbb{C}$-linear, and that the R-bilinear form*

   $$f(x, y) = g(x, y) - g(y, x)$$

   is real-valued on $E \times E$. Show that there exists a hermitian form h on E and a symmetric $\mathbb{C}$-bilinear form ψ on E such that $2ig = h + \psi$. Show that h and ψ are uniquely determined.