代数 1 H 班 作业 12

2023 年 8 月 3 日

题 1. *Prove that a diagonalizable linear operator is also diagonalizable on any invariant subspace.*

题 2. 请分类 $\mathbb{Q}$ 上的一维对称双线性型的同构类。

题 3. 请证明 $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 和 $A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ 做为 *Gram* 矩阵定义在 $\mathbb{Q}^2$ 上的两个对称双线性型不同构。

题 4. *Artin Chapter 8, 4.12.*

题 5. *Artin Chapter 8, 4.13.*

题 6. *Artin Chapter 8, 6.8.*

题 7. *Artin Chapter 8, 6.12.*

题 8. *Artin Chapter 8, 6.19.*

题 9. *Let V be a finite dimensional vector space over $\mathbb{C}$ with a positive Hermitian form $\langle , \rangle$. (or in other words, V is a Hermitian space.)*

1. *Prove that a Hermitian operator T on V defines a Hermitian form by $h(v, w) = \langle Tv, w \rangle$.*

2. *Prove that h is positive definite if and only if the eigenvalues of T are positive. We also call T positive definite in this case.*

3. *Prove that h is positive definite if and only if there is a Hermitian operator P with positive eigenvalues such that $T = P^2$, and such operator P is uniquely determined by T. We denote by $P = T^{\frac{1}{2}}$.*

4. *Prove that for any invertible linear operator Q on V, the linear operator $T = Q^*Q$ is hermitian, and positive definite.*

5. *Following the previous notation, prove that $U = QP^{-1}$ is a unitary operator. This proves the polar decomposition for invertible operator $Q = UP$ with P positive definite Hermitian and U unitary, here U and P are uniquely determined by Q. In other words, the group $GL(n, \mathbb{C})$ is the "product" of $U(n)$ and a "positive cone".*

题 10 (Courant-Fischer-Weyl min-max principle)**.** 设 $(E, \langle\cdot,\cdot\rangle)$ 是一个 n 维厄米空间 *(即复 $\langle\cdot,\cdot\rangle$ 正定)*. 假设 T 是一个 *hermitian* 变换, 有实特征值 $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. 证明 *hermitian* 变换的特征值可以由如下的 *Min-Max* 方法给出.

$$\lambda_k = \min\{\max\{\langle T(x), x\rangle : x \perp W_k, |x| = 1\} : W_k \subset E \text{ 子空间}, \dim W_k = k-1\}$$

这里先固定 $k-1$ 维子空间 W_k, 取出对应的最大值

$$\max\{\langle T(x), x\rangle : x \perp W_k, |x| = 1\}.$$

然后让 W_k 取遍 $k-1$ 维子空间, 取出这些值中的最小值。

题 11 (Cauchy's interlace theorem)**.** *Let A be a hermitian $n \times n$ matrix, and B be a $m \times m$ principal submatrix of A, for some $m < n$. If the eigenvalues of A are $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, and the eigenvalues of B are $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_m$, then for all $1 \leqslant i \leqslant m$*

$$\lambda_i \geqslant \mu_i \geqslant \lambda_{i+n-m}.$$

(Hint: use the previous Courant-Fischer-Weyl min-max principle)

题 12 (Sylvester's criterion)**.** *Prove that a Hermitian matrix is positive-definite if and only if the leading principal minors (顺序主子式) are positive.* (*Hint: Use Cauchy's interlace theorem*)

题 13 (Lang Chapter XV, exercise 5)**.** *Let E be a finite-dimensional space over the complex numbers, with a positive definite hermitian form. Let S be a set of ($\mathbb{C}$-linear) endomorphisms of E having no invariant subspace except* 0 *and E. (This means that if F is a subspace of E and $BF \subset F$ for all*

$B \in S$, then $F = 0$ or $F = E$.) Let A be a hermitian map of E into itself such that $AB = BA$ for all $B \in S$. Show that $A = \lambda I$ for some real number λ. [Hint: Show that there exists exactly one eigenvalue of A. If there were two eigenvalues, say $\lambda_1 \neq \lambda_2$, one could find two polynomials f and g with real coefficients such that $f(A) \neq 0, g(A) \neq 0$ but $f(A)g(A) = 0$. Let F be the kernel of $g(A)$ and get a contradiction.]

题 14 (Lang Chapter XV, exercise 6)**.** *Let E be as in previous exercise. Let T be a $\mathbb{C}$-linear map of E into itself. Let*

$$A = \frac{1}{2}\left(T + T^*\right).$$

Show that A is hermitian. Show that T can be written in the form $A + iB$ where A, B are hermitian, and are uniquely determined.

题 15 (Lang Chapter XV, exercise 7)**.** *Let S be a commutative set of $\mathbf{C}$-linear endomorphisms of E having no invariant subspace unequal to 0 or E. Assume in addition that if $B \in S$, then $B^* \in S$. Show that each element of S is of type αI for some complex number α. [Hint: Let $B_0 \in S$. Let*

$$A = \frac{1}{2}\left(B_0 + B_0^*\right).$$

Show that $A = \lambda I$ for some real λ.]