代数 1 H 班 作业 13

2023 年 8 月 3 日

题 1 (Hermann Weyl's question)**.** *Let A and B be two 2×2 Hermitian matrices with eigenvalues $\lambda_1 \leq \lambda_2$ and $\mu_1 \leq \mu_2$. Please find the possible range of eigenvalues for $A + B$ in terms of λ_i, μ_i. (感兴趣的同学可以思考，不用交，How about 3×3 matrices? The general case was solved by Allen Knutson and Terence Tao. https://www.ams.org/notices/200102/fea-knutson.pdf)*

题 2. *Let A_i be a rotation with angle θ_i in 3-dimensional Euclidean space fixing origin. Prove that the composition $A_1 A_2$ is still a rotation with angle α. Find the possible range for α in terms of θ_i.*

题 3. *Let $(F^\times)^2$ be nonzero elements in F with square roots and $F^\times/(F^\times)^2$ be the quotient group. Let G be the Gram matrix for a nondegenerate bilinear form g on V. Prove that $g \mapsto \det G \in F^\times/(F^\times)^2$ only depends on g. We will denote this by determinant* $\det(g)$

题 4. *Let F be a field with characteristic not equal to 2 and V be a vector space over F with non degenerate symmetric bilinear form g. Prove that the orthogonal sum of (V, g) and $(V, -g)$ is isometric to a hyperbolic space.*

题 5. *Let F be a field with characteristic not equal to 2 and V be a vector space over F with symmetric bilinear form g. Assume $\dim V = 2$. Prove that (V, g) is hyperbolic if and only if $\det g = -1 \in F^\times/(F^\times)^2$.*

题 6. *Let p be an odd prime number. Classify two dimensional symmetric bilinear forms on $(\mathbb{F}_p)^2$.*

题 7. *Prove that the dimension of maximal totally isotropic subspace in* (V, g) *is a constant. Here totally isotropic subspace* W *being maximal means if there is totally isotropic subspace* W' *containing* W*, then* $W' = W$.

题 8. *Prove that a non-degenerate bilinear form* (V, g) *is orthogonal sum of a hyperbolic space and an anisotropic subspace* W*, and the isometry class of* W *is uniquely determined by* (V, g).

题 9. *Define an alternating form* ω *on* $W = V \oplus V^*$ *similar as hyperbolic space by* $\omega((v_1, f_1), (v_2, f_2)) = f_1(v_2) - f_2(v_1)$ *for any* $v_i \in V, f_i \in V^*$. *The group* $Sp(W)$ *is the isometry group of* (W, ω).

1. *Show that the linear isomorphisms of* W *induce isometries of* (W, ω). *Use this to find an injective group homorphism* $GL(V) \to Sp(W)$.
2. *Prove that when dimension of* W *is 2, then* $SL(W) \cong Sp(W)$.
3. *Let* $W^* = (V \oplus V^*)^* = V^* \oplus V$ *equip with similar alternating form induced by* $(V^*)^* = V$. *Show that if* $F \in Sp(W)$*, then* $F^* \in Sp(W^*)$.
4. *Let* $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ *and* $Sp(2n, F) = \{A \in GL(2n, F) \mid A^T J A = J\}$. *Prove that if* $A \in Sp(2n, F)$*, so is* A^T.
5. *We call a subspace* $L \subset W$ *such that* $\omega\mid_L = 0$ *an isotropic subspace. Find all the possible dimensions of maximal isotropic subspaces. Such subspaces are called Lagrangian subspaces.*

题 10. *Prove the following identities for Pffafian. For a* $2n \times 2n$ *skew-symmetric matrix* A *and* $\lambda \in F$

$$\operatorname{pf}\left(A^{\mathrm{T}}\right) = (-1)^n \operatorname{pf}(A).$$
$$\operatorname{pf}(\lambda A) = \lambda^n \operatorname{pf}(A).$$
$$\operatorname{pf}\left(A^{2m+1}\right) = (-1)^{nm} \operatorname{pf}(A)^{2m+1}.$$

Prove the Pffafian is an invariant function on the space of skew-symmetric matrices under the orthogonal group action. In other words, $\operatorname{pf}(A) = \operatorname{pf}(BAB^{-1})$ *for any* $B \in O(2n)$. *(感兴趣同学可以思考，不用交，所有关于反对陈矩阵元素的多项式函数，如果在* $O(2n)$ *作用下不变，则一定是写作特征多项式的系数以及 Pffafian 的多项式。对奇数维呢？)*

题 11. *Artin Chapter 8, 5.4*

题 12. *Artin Chapter 8, 6.4*

题 13. *Artin Chapter 8, 8.1*

题 14. *Artin Chapter 8, 8.3*